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# Embedding of theories with SU(2|4) symmetry into the plane wave matrix model

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Abstract: We study theories with $SU(2|4)$ symmetry, which include the plane wave matrix model, $2+1$ SYM on $R \times S^2$ and $\mathcal{N}=4$ SYM on $R \times S^3/Z_k$. All these theories possess many vacua. From Lin-Maldacena's method which gives the gravity dual of each vacuum, it is predicted that the theory around each vacuum of $2+1$ SYM on $R \times S^2$ and $\mathcal{N}=4$ SYM on $R \times S^3/Z_k$ is embedded in the plane wave matrix model. We show this directly on the gauge theory side. We clearly reveal relationships among the spherical harmonics on $S^3$, the monopole harmonics and the harmonics on fuzzy spheres. We extend the compactification (the T-duality) in matrix models a la Taylor to that on spheres.

Keywords: Gauge-gravity correspondence, Supersymmetric gauge theory.





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## Contents

## 1. Introduction

The gauge/gravity (string) correspondence is one of the most important concepts in studying nonperturbative aspects of string theory and gauge theories. An exhaustively investigated example is the AdS/CFT correspondence [1 – 3]. Recently, Lin and Maldacena proposed the gauge/gravity correspondence for theories with $SU(2|4)$ symmetry [4], which include on the gauge theory side the plane wave matrix model (PWMM) [5], $2+1$ super Yang Mills on $R \times S^2$ ($\mathrm{SYM}_{R\times S^2}$) [6] and $\mathcal{N}=4$ super Yang Mills on $R \times S^3/Z_k$ ($\mathrm{SYM}_{R\times S^3/Z_k}$). These theories share the common feature that they have many vacua, a mass gap and a discrete energy spectrum. Lin and Maldacena developed a unified method for providing the gravity dual of each vacuum of these theories. This method is an extension of the so-called bubbling AdS geometry [7].

From Lin-Maldacena's method, it is predicted that the theory around each vacuum of $\mathrm{SYM}_{R\times S^2}$ and $\mathrm{SYM}_{R\times S^3/Z_k}$ is embedded in PWMM. In this paper, we prove this prediction for every vacuum of $\mathrm{SYM}_{R\times S^2}$ and the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$. Our results do not only serve as a nontrivial check of the gauge/gravity correspondence for the theories with $SU(2|4)$ symmetry, but they are also interesting in the following aspects. First, we extend the compactification (the T-duality) in matrix models a la Taylor [8] to that on spheres. We realize $S^3/Z_k$ as a $U(1)$ bundle on $S^2$ in matrices. Second, we clearly reveal relationships among various spherical harmonics: the spherical harmonics on $S^3$, the monopole harmonics developed by Wu, Yang and others [9 – 12] and the harmonics on a set of concentric fuzzy spheres with different radii [13 – 15]. We give an alternative understanding and a generalization of topologically nontrivial configurations and their topological charges on fuzzy spheres studied in [16 – 20]. Our results would shed light on problems of describing curved space [21] and topological invariants in matrix models [22 – 24]. In what follows, we review known facts on the gauge theory side and the gravity side of the theories with $SU(2|4)$ symmetry as well as describe our strategy and the organization of this paper.

In [4], PWMM, $\mathrm{SYM}_{R\times S^2}$ and $\mathrm{SYM}_{R\times S^3/Z_k}$ were defined by truncations of $\mathcal{N}=4$ SYM on $R \times S^3$ ($\mathrm{SYM}_{R\times S^3}$) as follows. $\mathrm{SYM}_{R\times S^3}$ has the superconformal symmetry $SU(2,2|4)$, whose bosonic subgroup is $SO(2,4) \times SO(6)$, where $SO(2,4)$ is the conformal group in 4 dimensions and $SO(6)$ is the R-symmetry. $SO(2,4)$ has a subgroup $SO(4)$ that is the isometry of the $S^3$ on which the theory is defined. $SO(4)$ is identified with $SU(2) \times \tilde{SU}(2)$, where we have marked one of two $SU(2)$'s with a tilde to focus on it. The above theories are obtained by dividing the original $\mathrm{SYM}_{R\times S^3}$ by subgroups of $\tilde{SU}(2)$. Dividing it by full $\tilde{SU}(2)$ gives rise to PWMM. Indeed this fact was first found in [25].[1] Dividing $\mathrm{SYM}_{R\times S^3}$ by $Z_k$ gives rise to $\mathrm{SYM}_{R\times S^3/Z_k}$. In a coordinate system of $S^3$ defined in appendix A, this corresponds to making an identification $(\theta,\phi,\psi) \sim (\theta,\phi,\psi+\frac{4\pi}{k})$. The $k \to \infty$ limit of $\mathrm{SYM}_{R\times S^3/Z_k}$ is nothing but $\mathrm{SYM}_{R\times S^2}$. That is, $\mathrm{SYM}_{R\times S^2}$ is obtained

[1] We make a remark on a relation of PWMM with a supersymmetric quantum mechanics that is given by the dimensional reduction of 10D $\mathcal{N}=1$ SYM to $1+0$ dimensions. General mass deformation of this quantum mechanics which preserves all supersymmetries was studied in [26], and it was recently shown in [27] that the deformation is unique and gives PWMM.



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by dividing $\mathrm{SYM}_{R\times S^3}$ by $U(1)$, in other words, by dimensionally reducing $\mathrm{SYM}_{R\times S^3}$ or $\mathrm{SYM}_{R\times S^3/Z_k}$ in the $\psi$ direction. In [6], the trivial vacuum of $\mathrm{SYM}_{R\times S^2}$ was obtained by removing fuzziness of fuzzy spheres in a vacuum of PWMM. By viewing this procedure inversely, one finds that PWMM is obtained as a dimensional reduction of $\mathrm{SYM}_{R\times S^2}$. It can be said that we achieve 'inverse' of these dimensional reductions in this paper, keeping the philosophy of [28] in mind: we obtain $\mathrm{SYM}_{R\times S^3/Z_k}$ from $\mathrm{SYM}_{R\times S^2}$ and $\mathrm{SYM}_{R\times S^2}$ from PWMM. In section 2.1, we review these dimensional reductions.

The vacua of PWMM are characterized by configuration of concentric membrane fuzzy spheres [5]. The vacua of $\mathrm{SYM}_{R\times S^2}$ are labeled by monopole charges and unbroken gauge group [6, 4]. The vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$ are parameterized by the holonomy along nontrivial generator of $\pi_1(S^3/Z_k)$ [4]. In section 2.2, we review these facts, and we clarify correspondence between the holonomy parameterizing the vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$ with $k\to\infty$ and the monopole charges and the unbroken gauge group labeling the vacua of $\mathrm{SYM}_{R\times S^2}$.

On the gravity side, Lin and Maldacena reduced the problem of finding a supergravity solution dual to each vacuum of the above theories to the problem of finding an axially symmetric solution to the 3-dimensional Laplace equation for the electrostatic potential, where the boundary condition involves charged conducting disks and a background potential. Each theory is specified by a background potential and each vacuum is specified by a configuration of charged conducting disks. In section 3.1, we review Lin-Maldacena's method and the one-to-one correspondences between the configurations of charged conducting disks and the vacua. In particular, by using the correspondence described in section 2.2, we clarify the one-to-one correspondence between the configurations of charged conducting disks and the monopole charges and the unbroken gauge group labeling the vacua of $\mathrm{SYM}_{R\times S^2}$.

In section 3.2, from the one-to-one correspondences between the configurations of charged conducting disks and the vacua, we obtain the following two predictions about relations between the vacua of different gauge theories: if the gauge/gravity correspondence for the theories with $SU(2|4)$ symmetry is valid, 1) the theory around each vacuum of $\mathrm{SYM}_{R\times S^2}$ is embedded in PWMM and 2) the theory around each vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ is embedded in $\mathrm{SYM}_{R\times S^2}$. More precisely, 1) the theory around each vacuum of $\mathrm{SYM}_{R\times S^2}$ is equivalent to the theory around a certain vacuum of PWMM and 2) the theory around each vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ is equivalent to the theory around a certain vacuum of $\mathrm{SYM}_{R\times S^2}$ with a periodicity imposed. In [6], the prediction 1) for the trivial vacuum of $\mathrm{SYM}_{R\times S^2}$ was already shown as mentioned above, and its consistency with the gravity duals was recently shown in [29]. The prediction 1) for some nontrivial vacua of $\mathrm{SYM}_{R\times S^2}$ was also suggested in [6, 30]. We give a complete proof of the prediction 1) for generic nontrivial vacua of $\mathrm{SYM}_{R\times S^2}$ in this paper. Combining the predictions 1) and 2) leads to a remarkable statement that the theory around every vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ and $\mathrm{SYM}_{R\times S^2}$ is embedded in PWMM.

In order to prove the predictions, we make harmonic expansions for the theories around various vacua. We use the spherical harmonics on $S^3$, the monopole harmonics on $S^2$ and the harmonics on a set of fuzzy spheres with different radii, which we call the fuzzy sphere harmonics. In section 4, as a preparation for the proofs, we describe properties of



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these harmonics. In section 4.1, we recall the properties of the spherical harmonics on $S^3$ summarized in [31] and add some new results. In section 4.2, we generalize the results on the monopole harmonics in [9 – 12] and reveal relationship between the monopole harmonics and the spherical harmonics on $S^3$. In section 4.3, we study the fuzzy sphere harmonics, which is an appropriate basis for the vector space of rectangular matrices [13 – 15]. We further develop the works [13 – 15]: we consider general spin $S$ fuzzy sphere harmonics and derive various formula about them, and furthermore we clearly reveal their relationship with the monopole harmonics. It is well known [32 – 34] that a basis for the vector space of square matrices is the harmonics on a fuzzy sphere and is regarded as a regularization of the ordinary spherical harmonics on $S^2$, where the size of matrices plays a role of an ultraviolet cut-off for the angular momentum. Analogously, a basis for the vector space of rectangular matrices is the fuzzy sphere harmonics and is regarded as a regularization of the monopole harmonics, where the size of matrices plays a role of an ultraviolet cut-off while a half of the difference between the numbers of raws and columns is fixed and identified with the monopole charge.

By using the results in sections 4.2 and 4.3, we prove the prediction 1) in section 5.1. In section 5.2, we comment on a relation of our result in section 5.1 with the works [19, 20]. In section 6.1, by using the results in sections 4.1 and 4.2 and the mode expansion around the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ performed in [31], we prove the prediction 2) for the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$. Following the suggestion given by the gravity side, we consider a configuration of matrices in $\mathrm{SYM}_{R\times S^2}$ with a periodicity and recover the $\psi$ direction by 'T-duality'. This is an extension of the compactification (the T-duality) in matrix models a la Taylor to that on spheres, where $S^3/Z_k$ is realized as a nontrivial $S^1$ fibration over $S^2$ in matrices rather than a direct product. In section 6.2, we combine the predictions 1) and 2) and make some comments on construction of $S^3$ in terms of three matrices.

Section 7 is devoted to summary and discussion. Some details are gathered in appendices.

## 2. Theories with $SU(2|4)$ symmetry

In this section, we review the gauge theory side of the theories with $SU(2|4)$ symmetry with some new insights. In section 2.1, starting with $\mathrm{SYM}_{R\times S^3}$ or $\mathrm{SYM}_{R\times S^3/Z_k}$, we first obtain $\mathrm{SYM}_{R\times S^2}$ by a dimensional reduction. After rewriting it using a 3-dimensional notation, we again make a dimensional reduction for it to obtain PWMM. We fix our notation in the above process. In section 2.2, we classify vacua of the theories with $SU(2|4)$ symmetry. In particular, we clarify correspondence between the vacua of $\mathrm{SYM}_{R\times S^2}$ and the vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$ with the $k\to\infty$ limit.

### 2.1 Dimensional reductions from $\mathcal{N}=4$ SYM on $R\times S^3$

We start with $\mathrm{SYM}_{R\times S^3}$ [38 – 41]. Here the gauge group is $U(N)$ and the radius of $S^3$ is fixed to $\frac{2}{\mu}$. Borrowing the ten-dimensional notation, we can write down the action as

follows:

$$S_{R\times S^3} = \frac{1}{g^2_{R\times S^3}} \int dt \frac{d\Omega_3}{(\mu/2)^3} \mathrm{Tr} \left( -\frac{1}{4} F_{ab} F^{ab} - \frac{1}{2} D_a X_m D^a X_m - \frac{1}{12} \hat{R} X_m^2 \right.$$
$$\left. -\frac{i}{2} \bar{\lambda} \Gamma^a D_a \lambda - \frac{1}{2} \bar{\lambda} \Gamma^m [X_m, \lambda] + \frac{1}{4} [X_m, X_n]^2 \right), \quad (2.1)$$

where $a$ and $b$ are the (3+1)-dimensional local Lorentz indices and run from 0 to 3, and $m$ runs from 4 to 9. $\Gamma^a$ and $\Gamma^m$ are the 10-dimensional gamma matrices, which satisfy

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \qquad \{\Gamma^m, \Gamma^n\} = 2\delta^{mn}, \quad (2.2)$$

where $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$. $\lambda$ is the Majorana-Weyl spinor in 10 dimensions, which satisfies

$$C_{10} \bar{\lambda}^T = \lambda, \quad \Gamma^{11} \lambda = \lambda, \quad (2.3)$$

where $C_{10}$ is the charge conjugation matrix. $\hat{R}$ is the scalar curvature of $S^3$ which is equal to $\frac{3\mu^2}{2}$. The field strength and the covariant derivatives take the form

$$F_{ab} = \nabla_a A_b - \nabla_b A_a - i[A_a, A_b],$$
$$D_a X_m = \nabla_a X_m - i[A_a, X_m], \quad D_a \lambda = \nabla_a \lambda - i[A_a, \lambda], \quad (2.4)$$

where

$$\nabla_a A_b = e_a^\mu (\partial_\mu A_b + \omega_{\mu b}{}^c A_c), \quad \nabla_a X_m = e_a^\mu \partial_\mu X_m, \quad \nabla_a \lambda = e_a^\mu (\partial_\mu \lambda + \frac{1}{4} \omega_\mu^{bc} \Gamma_{bc} \lambda). \quad (2.5)$$

In appendix A, we list the metric, the vierbeins and the spin connections for $R \times S^3$ used in this paper. In this metric,

$$\int d\Omega_3 = \frac{1}{8} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi \sin\theta, \quad (2.6)$$

so that $\int d\Omega_3 1 = 2\pi^2$.

$\mathrm{SYM}_{R\times S^3/Z_k}$ is obtained by identifying the value at $(\theta, \phi, \psi)$ with that at $(\theta, \phi, \psi + \frac{4\pi}{k})$ for all the fields in $\mathrm{SYM}_{R\times S^3}$. The relation between the coupling constant of $\mathrm{SYM}_{R\times S^3/Z_k}$ and that of $\mathrm{SYM}_{R\times S^3}$ is given by

$$g^2_{R\times S^3} = k g^2_{R\times S^3/Z_k}. \quad (2.7)$$

The $k \to \infty$ limit of this procedure can be regarded as a dimensional reduction. This dimensional reduction with a redefinition of the gauge fields gives rise to $\mathrm{SYM}_{R\times S^2}$.

In order to obtain $\mathrm{SYM}_{R\times S^2}$, we make following replacements:

$$A = A_0 dt + A_\theta d\theta + A_\phi d\phi + A_\psi d\psi \ \to A_0 dt + A_\theta d\theta + (A_\phi + \frac{1}{\mu} \cos\theta \Phi) d\phi + \frac{1}{\mu} \Phi d\psi, \quad (2.8)$$

We also assume that all the fields are independent of $\psi$. Then, using the metric, the dreibeins and the spin connections for $R \times S^2$ listed in appendix A, it is easy to see that

(2.1) is reduced to an action on $R \times S^2$. For instance, the space components of the gauge field strength are reduced to quantities on $R \times S^2$ as

$$F_{12} \ \rightarrow \ F_{12} - \mu\Phi, \quad F_{13} \ \rightarrow \ D_1\Phi, \quad F_{23} \ \rightarrow \ D_2\Phi. \tag{2.9}$$

The final result is

$$\begin{aligned} S_{R\times S^2} = \frac{1}{g^2_{R\times S^2}} \int dt \frac{d\Omega_2}{\mu^2} \mathrm{Tr} \Big( & -\frac{1}{4} F_{a'b'} F^{a'b'} - \frac{1}{2} D_{a'}\Phi D^{a'}\Phi - \frac{\mu^2}{2}\Phi^2 + \mu F_{12}\Phi \\ & -\frac{1}{2} D_{a'} X_m D^{a'} X_m - \frac{\mu^2}{8} X_m^2 + \frac{1}{4}[X_m, X_n]^2 + \frac{1}{2}[\Phi, X_m]^2 \\ & -\frac{i}{2}\bar{\lambda}\Gamma^{a'} D_{a'}\lambda + \frac{i\mu}{8}\bar{\lambda}\Gamma^{123}\lambda - \frac{1}{2}\bar{\lambda}\Gamma^3[\Phi,\lambda] - \frac{1}{2}\bar{\lambda}\Gamma^m[X_m,\lambda] \Big) \end{aligned} \tag{2.10}$$

where $a'$ and $b'$ are the $(2+1)$-dimensional local Lorentz indices and run from 0 to 2. The radius of $S^2$ is fixed to $\frac{1}{\mu}$ and

$$\int d\Omega_2 = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta, \tag{2.11}$$

so that $\int d\Omega_2 1 = 4\pi$. When $\mathrm{SYM}_{R\times S^2}$ is identified with the $k \rightarrow \infty$ limit of $\mathrm{SYM}_{R\times S^3/Z_k}$, the coupling constant $g_{R\times S^2}$ is expressed as

$$g^2_{R\times S^2} = \lim_{k\rightarrow\infty} \frac{k\mu g^2_{R\times S^3/Z_k}}{4\pi}, \tag{2.12}$$

so that $k g^2_{R\times S^3/Z_k}$ must be fixed in the $k \rightarrow \infty$ limit. This relation will be used in comparison with the gravity duals in section 3.1. (2.10) is $\mathrm{SYM}_{R\times S^2}$ obtained in [6].

For later convenience, we rewrite (2.10) using the 3-dimensional flat space notation, which is represented by the orthogonal coordinates system $(x_1, x_2, x_3)$ or the polar coordinates system $(r, \theta, \phi)$. We introduce the flat space nabla

$$\vec{\partial} = \vec{e}_i \partial_i = \vec{e}_r \partial_r + \vec{e}_\theta \frac{1}{r}\partial_\theta + \vec{e}_\phi \frac{1}{r\sin\theta}\partial_\phi, \tag{2.13}$$

where $\vec{e}_i \ \ (i = 1, 2, 3)$ are the unit vectors of $x_i$ directions, and $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_\phi$ are the unit vectors of the $r$, $\theta$ and $\phi$ directions, respectively. In the followings, the $r$-derivative in $\vec{\partial}$ does not contribute and $r$ in $\vec{\partial}$ is fixed to $\frac{1}{\mu}$. We construct a 3-dimensional vector from $A_\theta$ and $A_\phi$ as

$$\vec{A} = \mu A_\theta \vec{e}_\theta + \frac{\mu}{\sin\theta} A_\phi \vec{e}_\phi, \tag{2.14}$$

and define a vector,

$$\vec{\Gamma} = \Gamma^i \vec{e}_i. \tag{2.15}$$

We make a unitary transformation for the fermion,

$$\lambda \ \rightarrow \ e^{\frac{\pi}{4}\Gamma_{12}} e^{\frac{\theta}{2}\Gamma_{31}} e^{\frac{\phi}{2}\Gamma_{12}} \lambda. \tag{2.16}$$

Then, it is easy to see the transformation of the following two terms:

$$\mathrm{Tr}\left(-\frac{i}{2}\bar{\lambda}\Gamma^{a'}D_{a'}\lambda\right) \rightarrow \mathrm{Tr}\left(-\frac{i}{2}\bar{\lambda}\Gamma^{0}D_{0}\lambda - \frac{i}{2}\bar{\lambda}\vec{\Gamma}\cdot(\vec{e}_r\times\vec{D})\lambda - \frac{i\mu}{2}\bar{\lambda}\Gamma^{123}\lambda\right), \quad (2.17)$$

$$\mathrm{Tr}\left(-\frac{1}{2}\bar{\lambda}\Gamma^{3}[\Phi,\lambda]\right) \rightarrow \mathrm{Tr}\left(-\frac{1}{2}\bar{\lambda}\vec{\Gamma}\cdot\vec{e}_r[\Phi,\lambda]\right). \quad (2.18)$$

where $\vec{D} = \vec{\partial} - i[\vec{A},\ ]$. The other terms including the fermion are unchanged. Note that the last term on the righthand side of (2.17) shifts the coefficient of the fermion mass term. In order to rewrite the bosonic part, we define the following quantities:

$$\begin{aligned}
&\vec{Y} = \vec{e}_r\Phi + \vec{e}_r\times\vec{A},\\
&\vec{L}^{(0)} = -i\mu^{-1}\vec{e}_r\times\vec{\partial},\\
&\vec{\mathcal{Z}} = \mu\vec{Y} + i(\mu\vec{L}^{(0)}\times\vec{Y} - \vec{Y}\times\vec{Y}),\\
&\vec{\mathcal{L}} = \mu\vec{L}^{(0)} - [\vec{Y},\ ].
\end{aligned} \quad (2.19)$$

$\vec{\mathcal{Z}}$ is evaluated as

$$\vec{\mathcal{Z}} = (-\mu\Phi + F_{12})\vec{e}_r + D_1\Phi\vec{e}_\theta + D_2\Phi\vec{e}_\phi. \quad (2.20)$$

Finally, we obtain

$$\begin{aligned}
S_{R\times S^2} = \frac{1}{g^2_{R\times S^2}}\int dt\frac{d\Omega_2}{\mu^2}\mathrm{Tr}\Big(&\frac{1}{2}(D_0\vec{Y} - i\mu\vec{L}^{(0)}A_0)^2 - \frac{1}{2}\vec{\mathcal{Z}}^2 + \\
&+\frac{1}{2}(D_0X_m)^2 + \frac{1}{2}(\vec{\mathcal{L}}X_m)^2 - \frac{\mu^2}{8}X_m^2\\
&+\frac{1}{4}[X_m,X_n]^2 - \frac{i}{2}\bar{\lambda}\Gamma^0D_0\lambda + \frac{1}{2}\bar{\lambda}\vec{\Gamma}\cdot\vec{\mathcal{L}}\lambda - \frac{3i\mu}{8}\bar{\lambda}\Gamma^{123}\lambda - \frac{1}{2}\bar{\lambda}\Gamma^m[X_m,\lambda]\Big).
\end{aligned} \quad (2.21)$$

It is now easy to obtain PWMM. We dimensionally reduce (2.21) to $1+0$ dimensions by dropping $\vec{\partial}$. The result is

$$\begin{aligned}
S_{PW} = \frac{1}{g^2_{PW}}\int\frac{dt}{\mu^2}\ \mathrm{Tr}\Big(&\frac{1}{2}(D_0Y_i)^2 - \frac{1}{2}(\mu Y_i - \frac{i}{2}\epsilon_{ijk}[Y_j,Y_k])^2 + \frac{1}{2}(D_0X_m)^2 - \frac{\mu^2}{8}X_m^2\\
&+\frac{1}{2}[Y_i,X_m]^2 + \frac{1}{4}[X_m,X_n]^2 - \frac{i}{2}\bar{\lambda}\Gamma^0D_0\lambda - \frac{3i\mu}{8}\bar{\lambda}\Gamma^{123}\lambda +\\
&-\frac{1}{2}\bar{\lambda}\Gamma^i[Y_i,\lambda] - \frac{1}{2}\bar{\lambda}\Gamma^m[X_m,\lambda]\Big),
\end{aligned} \quad (2.22)$$

where $4\pi g^2_{PW} = g^2_{R\times S^2}$. In appendix B, we show that this is indeed equivalent to the action of PWMM used in the literature.

In appendix C, we describe the supersymmetry transformations of all the theories. In appendix A, we rewrite the actions (2.1), (2.21) and (2.22) in terms of the $SU(4)$ symmetric notation. We will make mode expansions for these $SU(4)$ symmetric forms of the actions in sections 5 and 6. In the remaining of the present paper, it is convenient to assume that the gauge groups of PWMM, $\mathrm{SYM}_{R\times S^2}$ and $\mathrm{SYM}_{R\times S^3/Z_k}$ are $U(\hat{N})$, $U(\tilde{N})$ and $U(N)$, respectively.

### 2.2 Nontrivial vacua

While $\mathrm{SYM}_{R\times S^3}$ has the unique trivial vacuum, $\mathrm{SYM}_{R\times S^3/Z_k}$ has many vacua. Those vacua are given by the space of flat connections on $S^3/Z_k$. The space is parameterized by the holonomy $U$ along nontrivial generator of $\pi_1(S^3/Z_k) = Z_k$ up to gauge transformations. $U$ satisfies $U^k = 1$, so that $U$ can be diagonalized as

$$U = \mathrm{diag}(\underbrace{e^{i\frac{2\pi}{k}\beta_1}, e^{i\frac{2\pi}{k}\beta_1}, \cdots, e^{i\frac{2\pi}{k}\beta_1}}_{N_1}, \underbrace{e^{i\frac{2\pi}{k}\beta_2}, e^{i\frac{2\pi}{k}\beta_2}, \cdots, e^{i\frac{2\pi}{k}\beta_2}}_{N_2}, \cdots, \underbrace{e^{i\frac{2\pi}{k}\beta_T}, e^{i\frac{2\pi}{k}\beta_T}, \cdots, e^{i\frac{2\pi}{k}\beta_T}}_{N_T}), \tag{2.23}$$

where all $\beta_s$ $(s = 1, \cdots, T,\ T \leq k)$ are different integers mod $k$, and $N_1 + \cdots + N_T = N$. The vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$ are parameterized by $U$ in (2.23). By applying the flat connection condition to the supersymmetry transformation (C.3), it is easy to see that these vacua preserve all 16 supercharges. In the vacuum (2.23), the gauge symmetry $U(N)$ is spontaneously broken to $U(N_1) \times U(N_2) \times \cdots \times U(N_T)$.

Next, let us discuss the vacua of $\mathrm{SYM}_{R\times S^2}$. The condition for the vacua of $\mathrm{SYM}_{R\times S^2}$ is obtained from the $k \to \infty$ limit of the condition for the vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$, which are given by the space of the flat connections on $R \times S^3/Z_k$. Then, it is seen from (2.9) that the condition for the vacua of $\mathrm{SYM}_{R\times S^2}$ is

$$\begin{aligned} &F_{12} - \mu\Phi = 0, \\ &D_1\Phi = D_2\Phi = 0. \end{aligned} \tag{2.24}$$

On the other hand, the condition for vacua derived from (2.21) is

$$\vec{\mathcal{Z}} = 0, \tag{2.25}$$

which is indeed equivalent to (2.24) as seen from (2.20). In order to solve the equations (2.24), we take a gauge in which $\Phi$ is diagonal. Then, the second equation in (2.24) implies that $\Phi$ is constant. We parameterize $\Phi$ as

$$\Phi = \frac{\mu}{2}\mathrm{diag}(\underbrace{\alpha_1, \alpha_1, \cdots, \alpha_1}_{N_1}, \underbrace{\alpha_2, \alpha_2, \cdots, \alpha_2}_{N_2}, \cdots, \underbrace{\alpha_T, \alpha_T, \cdots, \alpha_T}_{N_T}), \tag{2.26}$$

where all $\alpha_s$'s $(s = 1, \cdots, T)$ are different, and $N_1 + \cdots + N_T = \tilde{N}$. Then, it is seen from the second equation in (2.24) that $A_1$ and $A_2$ are block-diagonal, where the sizes of the blocks are $N_1, N_2, \cdots, N_T$. Using the remaining $U(N_1) \times U(N_2) \times \cdots \times U(N_T)$, we take a gauge in which $A_1 = 0$. Then, the first equation reduces to

$$\nabla_1 A_2 + \mu \cot\theta A_2 = \mu\Phi. \tag{2.27}$$

This equation can be easily solved by introducing patches on $S^2$ as

$$A_2 = \begin{cases} \tan\frac{\theta}{2}\,\Phi & \text{in region I} \\ -\cot\frac{\theta}{2}\,\Phi & \text{in region II} \end{cases}, \tag{2.28}$$

where the region I corresponds to $0 \leq \theta < \frac{\pi}{2} + \varepsilon$ while the region II corresponds to $\frac{\pi}{2} - \varepsilon < \theta \leq \pi$. To summarize, the solution to (2.24) is

$$\begin{aligned}
&\hat{\Phi} = \frac{\mu}{2}\mathrm{diag}(\underbrace{\alpha_1, \alpha_1, \cdots, \alpha_1}_{N_1}, \underbrace{\alpha_2, \alpha_2, \cdots, \alpha_2}_{N_2}, \cdots, \underbrace{\alpha_T, \alpha_T, \cdots, \alpha_T}_{N_T}), \\
&\hat{A}_1 = 0, \\
&\hat{A}_2 = \begin{cases} \tan\frac{\theta}{2}\,\hat{\Phi} & \text{in region I} \\ -\cot\frac{\theta}{2}\,\hat{\Phi} & \text{in region II} \end{cases}
\end{aligned} \tag{2.29}$$

Each diagonal element of $\hat{A}_1$ and $\hat{A}_2$ is the configuration of a monopole with magnetic charge $q_s = \frac{\alpha_s}{2}$. In the overlap of the regions I and II, the configurations in both patches are transformed each other by the gauge transformation given by

$$V_{I \to II} = \exp\left(i\frac{2}{\mu}\hat{\Phi}\phi\right). \tag{2.30}$$

It follows from the single-valuedness of $V_{I\to II}$ that all $\alpha_s$'s ($s = 1, \cdots, T$) in (2.29) are integers. This is nothing but Dirac's quantization condition for the monopole charges. One can understand this condition from a different point of view as follows. In the $k \to \infty$ limit, each vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ would reduce to a vacuum of $\mathrm{SYM}_{R\times S^2}$. As mentioned in the previous subsection, $S^3/Z_k$ is obtained by making an identification on $S^3$, $(\theta, \phi, \psi) \sim (\theta, \phi, \psi + \frac{4\pi}{k})$. A generator of $\pi_1(S^3/Z_k)$ is a non-contractible loop, $C : (\frac{\pi}{2}, 0, \psi)$ $\psi \in [0, \frac{4\pi}{k}]$. The holonomy along this loop is

$$U = P\exp\left[i\int_0^{\frac{4\pi}{k}} A_\psi d\psi\right]. \tag{2.31}$$

In the $k \to \infty$ limit, from (2.8), this reduces to

$$U = \exp\left[i\frac{4\pi}{k}\frac{1}{\mu}\Phi(\theta, \phi)\right]. \tag{2.32}$$

Substituting (2.26) into (2.32) yields

$$\begin{aligned}
U = \mathrm{diag}(&\underbrace{e^{i\frac{2\pi}{k}\alpha_1}, e^{i\frac{2\pi}{k}\alpha_1}, \cdots, e^{i\frac{2\pi}{k}\alpha_1}}_{N_1}, \underbrace{e^{i\frac{2\pi}{k}\alpha_2}, e^{i\frac{2\pi}{k}\alpha_2}, \cdots, e^{i\frac{2\pi}{k}\alpha_2}}_{N_2}, \cdots, \\
&\underbrace{e^{i\frac{2\pi}{k}\alpha_T}, e^{i\frac{2\pi}{k}\alpha_T}, \cdots, e^{i\frac{2\pi}{k}\alpha_T}}_{N_T}).
\end{aligned} \tag{2.33}$$

The condition $U^k = 1$ indeed implies that all $\alpha_s$'s ($s = 1, \cdots, T$) are integers. This consideration also clarifies correspondence between the vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$ with the $k \to \infty$ limit and the vacua of $\mathrm{SYM}_{R\times S^2}$. Using (C.2), it is easy to show that the vacua (2.29) preserve all 16 supercharges. In the vacuum (2.29), the gauge group $U(\tilde{N})$ is spontaneously broken to $U(N_1) \times U(N_2) \times \cdots \times U(N_T)$.

Finally, we discuss the vacua of PWMM. The condition for the vacua would be obtained by dropping the derivative in (2.25). The result is

$$\mu Y_i - \frac{i}{2}\epsilon_{ijk}[Y_j, Y_k] = 0. \tag{2.34}$$

This condition is also read off directly from (2.22). The general solution to the equation (2.34) is

$$Y_i = -\mu L_i, \tag{2.35}$$

where $L_i$ is a representation matrix for a $\hat{N}$-dimensional representation of $SU(2)$, which is in general reducible, and satisfies $[L_i, L_j] = i\epsilon_{ijk}L_k$. One can decompose it into irreducible pieces as

$$L_i = \begin{pmatrix} \overbrace{L_i^{[j_1]} \ddots L_i^{[j_1]}}^{N_1} & & & \\ & \overbrace{L_i^{[j_2]} \ddots L_i^{[j_2]}}^{N_2} & & \\ & & \ddots & \\ & & & \overbrace{L_i^{[j_T]} \ddots L_i^{[j_T]}}^{N_T} \end{pmatrix} \tag{2.36}$$

where $L_i^{[j_s]}$ $(s = 1, \cdots, T)$ stands for the $(2j_s + 1) \times (2j_s + 1)$ representation matrix for the spin $j_s$ representation of $SU(2)$ and satisfies

$$\begin{aligned} &[L_i^{[j_s]}, L_j^{[j_s]}] = i\epsilon_{ijk}L_k^{[j_s]}, \\ &(L_i^{[j_s]})^2 = j_s(j_s + 1)1_{2j_s+1}, \end{aligned} \tag{2.37}$$

and

$$(2j_1 + 1)N_1 + (2j_2 + 1)N_2 + \cdots + (2j_T + 1)N_T = \hat{N}. \tag{2.38}$$

The vacuum (2.36) can be interpreted as a set of coincident $N_s$ fuzzy spheres with the radius $\mu\sqrt{j_s(j_s + 1)}$ $(s = 1, \cdots, T)$, where all the fuzzy spheres are concentric. One can see from (C.1) that this vacuum preserves all 16 supercharges. In this vacuum, the gauge symmetry $U(\hat{N})$ is spontaneously broken to $U(N_1) \times U(N_2) \times \cdots \times U(N_T)$.

## 3. Gravity duals

In this section, we consider the gravity duals of the theories with $SU(2|4)$ symmetry. In section 3.1, we review the electrostatics problem that gives the gravity dual of each vacuum of these theories. In section 3.2, from relations between the configurations of conducting disks for the vacua, we obtain two predictions on relations between the vacua of different theories.

### 3.1 Electrostatics problem

It was shown in [4] that a general smooth solution of type IIA supergravity that preserves the $SU(2|4)$ symmetry is characterized by a single function $V(\rho, \eta)$ and takes the form

$$
\begin{aligned}
ds_{10}^2 &= \left(\frac{\ddot{V}-2\dot{V}}{-V''}\right)\left\{-4\frac{\ddot{V}}{\ddot{V}-2\dot{V}}dt^2+\frac{-2V''}{\dot{V}}(d\rho^2+d\eta^2)+4d\Omega_5^2+2\frac{V''\dot{V}}{\Delta}d\Omega_2^2\right\},\\
e^{4\phi} &= \frac{4(\ddot{V}-2\dot{V})^3}{-V''\dot{V}^2\Delta^2},\\
C_1 &= -\frac{2\dot{V}'\dot{V}}{\ddot{V}-2\dot{V}}dt,\\
F_4 &= dC_3, \quad C_3 = -4\frac{\dot{V}^2V''}{\Delta}dt\wedge d^2\Omega,\\
H_3 &= dB_2, \quad B_2 = \left(\frac{\dot{V}\dot{V}'}{\Delta}+\eta\right)d^2\Omega,\\
\Delta &= (\ddot{V}-2\dot{V})V''-(\dot{V}')^2,
\end{aligned}
\tag{3.1}
$$

where the dot and the prime stands for the derivatives with respect to $\log\rho$ and $\eta$, respectively. $V$ can be regarded as an electrostatic potential for an axially symmetric system with conducting disks and a background potential. $\rho$ is the distance from the center axis and $\eta$ is the coordinate in the direction along the center axis. $V$ is decomposed as $V = V_b(\rho,\eta)+v(\rho,\eta)$, where $V_b$ is the background potential, and $v$ is determined by a configuration of conducting disks. Each theory is specified by $V_b$ and each vacuum is specified by a configuration of conducting disks. The distance $d$ between two disks is proportional to the NS 5-brane charge, $d = \frac{\pi}{2}N_5$, while the electric charge $Q$ on a disk is proportional to the D2-brane charge, $Q = \frac{\pi^2}{8}N_2$.

The background potential for $\mathrm{SYM}_{R\times S^3/Z_k}$ is

$$
V_b = W(\rho^2-2\eta^2),
\tag{3.2}
$$

where $W = c/kg^2_{R\times S^3/Z_k}$ with $c$ a constant [4]. In this case, the system is periodic with respect to $\eta$ with the period $\frac{\pi}{2}k$, and the total NS 5-brane charge is $k$. One can concentrate a region $0\le\eta\le\frac{\pi}{2}k$, where one can place conducting disks at $\eta = 0, \frac{\pi}{2}, \cdots, \frac{\pi}{2}(k-1)$. For the vacuum (2.23), $T$ disks are located at $\eta_1 = \frac{\pi}{2}\beta_1, \eta_2 = \frac{\pi}{2}\beta_2, \cdots, \eta_T = \frac{\pi}{2}\beta_T$. The electric charges on these disks are equal to $\frac{\pi^2}{8}N_1, \frac{\pi^2}{8}N_2, \cdots, \frac{\pi^2}{8}N_T$, respectively. Figure 1 shows this configuration of conducting disks.

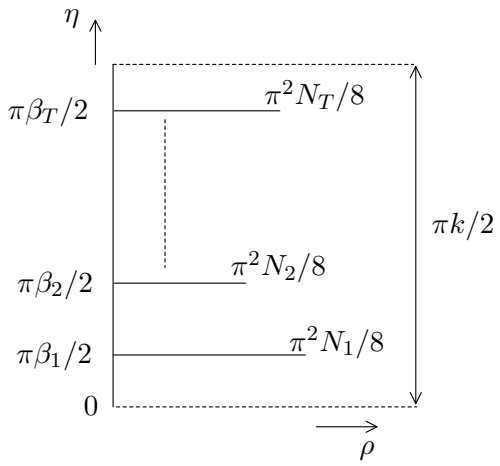


**Figure 1:** Configuration of conducting disks for (2.23)

$\mathrm{SYM}_{R\times S^2}$ corresponds to the $k\to\infty$ limit of $\mathrm{SYM}_{R\times S^3/Z_k}$. For $\mathrm{SYM}_{R\times S^2}$, the region of $\eta$ becomes infinite. The background potential for $\mathrm{SYM}_{R\times S^2}$ is given by

$$V_b = \tilde{W}(\rho^2 - 2\eta^2), \tag{3.3}$$

where $\tilde{W}$ is given by the $k\to\infty$ limit of $W$, so that $kg^2_{R\times S^3/Z_k}$ must be fixed. This is consistent with the result in the gauge theory side, and from (2.12) $\tilde{W}$ turns out to be $c\mu/4\pi g^2_{R\times S^2}$. By using the correspondence between the vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$ with the $k\to\infty$ limit and the vacua of $\mathrm{SYM}_{R\times S^2}$ seen in the previous subsection, it is easy to construct a configuration of conducting disks for each vacuum of $\mathrm{SYM}_{R\times S^2}$. For the vacuum (2.33), there are $T$ disks located at $\eta_1 = \frac{\pi}{2}\alpha_1, \eta_2 = \frac{\pi}{2}\alpha_2, \cdots, \eta_T = \frac{\pi}{2}\alpha_T$. The electric charges on these disks are equal to $\frac{\pi^2}{8}N_1, \frac{\pi^2}{8}N_2, \cdots, \frac{\pi^2}{8}N_T$, respectively. Figure 2 shows this configuration of conducting disks.

The background potential for PWMM is

$$V_b = \hat{W}(\rho^2\eta - \frac{2}{3}\eta^3), \tag{3.4}$$

where $\hat{W}$ is represented in terms of a certain function $h$ as [29]

$$\hat{W} = \frac{1}{g^2_{PW}} h(g^2_{PW}\hat{N}). \tag{3.5}$$

It was pointed out in [29] that the correspondence between the trivial vacuum of $\mathrm{SYM}_{R\times S^2}$ and a certain vacuum of PWMM shown in [6] is consistent with the gravity side only if the function $h$ approaches some constant $h_\infty$ at large values of its argument. Namely, this behavior of $h$ is true if the gauge/gravity correspondence for the theories with $SU(2|4)$ symmetry is valid. We assume this behavior, and we will use this assumption to obtain the

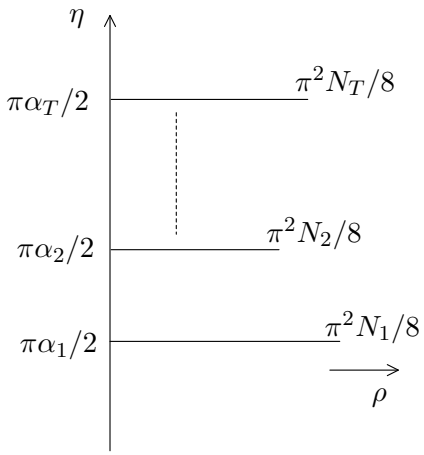


**Figure 2:** Configuration of conducting disks for (2.29)

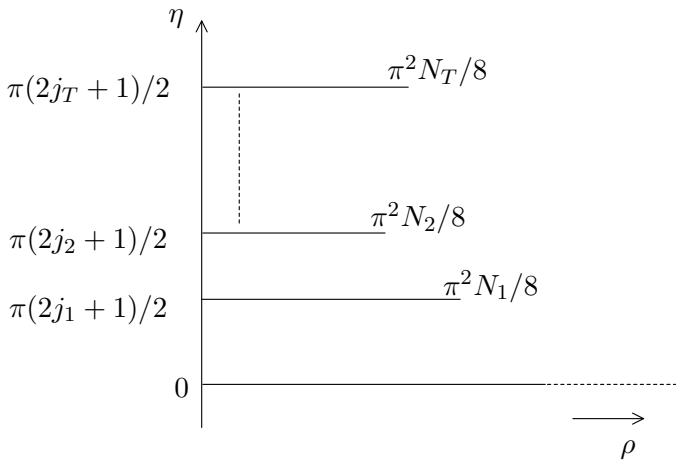


**Figure 3:** Configuration of conducting disks for (2.36)

prediction 1). In the case of PWMM, only the region $\eta \geq 0$ is meaningful. There is always a infinitely large disk sitting at $\eta = 0$. For the vacuum (2.36), there are $T$ disks other than this disk. They are located at $\eta_1 = \frac{\pi}{2}(2j_1+1), \eta_2 = \frac{\pi}{2}(2j_2+1), \cdots, \eta_T = \frac{\pi}{2}(2j_T+1)$. The electric charges on these disks are equal to $\frac{\pi^2}{8}N_1, \frac{\pi^2}{8}N_2, \cdots, \frac{\pi^2}{8}N_T$, respectively. Figure 3 shows this configuration of conducting disks.

### 3.2 Predictions on relations between vacua

We first consider a limit that transforms a vacuum of PWMM into a vacuum of $\mathrm{SYM}_{R\times S^2}$. Naively, by moving the infinitely large disk in a configuration for a vacuum of PWMM away

to infinity as in figure 4, one obtains a configuration of disks for a vacuum of $\mathrm{SYM}_{R\times S^2}$. This motivates us to take the following limit. We parameterize the positions of the disks for a vacuum of PWMM, which are proportional to the dimensions of representations of $SU(2)$ in the gauge theory, as

$$\begin{aligned} &2j_s+1 = N_0+\zeta_s, \\ &\eta_s = \eta_0+\tilde{\eta}_s, \\ &\eta_0 = \frac{\pi}{2}N_0, \quad \tilde{\eta}_s = \frac{\pi}{2}\zeta_s, \end{aligned} \tag{3.6}$$

where $N_0$ and $\zeta_s$ are integers. Under a shift $\eta \to \eta_0+\eta$, the background potential (3.4) is transformed as

$$V_b \ \to \ -\frac{2}{3}\hat{W}\eta_0^3 - 2\hat{W}\eta_0^2\eta + \hat{W}\eta_0(\rho^2-2\eta^2) + \hat{W}(\eta\rho^2 - \frac{2}{3}\eta^3) \tag{3.7}$$

The first and second terms on the righthand side do not contribute to the Laplace equation, the boundary condition for $V$ and the geometry. In the limit,

$$\eta_0 \to \infty, \quad \hat{W}\to 0, \quad \hat{W}\eta_0 = \tilde{W} = \text{fixed}, \tag{3.8}$$

the last term vanishes and only the third term survives resulting in the background potential for $\mathrm{SYM}_{R\times S^2}$. In the $T=1$ case, it was explicitly shown in [29] that the charge $Q_1$ can be fixed in this limit. It is reasonable to expect that all the charges $Q_s$'s ($s=1,\cdots,T$) can be fixed in this limit for generic $T$. Hence, the limit (3.8) indeed transforms the gravity dual of a vacuum of PWMM to the gravity dual of a vacuum of $\mathrm{SYM}_{R\times S^2}$ (See figure 4). This observation on the gravity side leads us to the prediction 1). Indeed, by using the relation between $\tilde{W}$ and $g_{R\times S^2}$ and the behavior of $h$ in $\hat{W}$ discussed in the previous subsection, we obtain the prediction 1) that on the gauge theory side the theory around the vacuum (2.36) of PWMM coincides with the theory around the vacuum (2.29) of $\mathrm{SYM}_{R\times S^2}$ with the identification $\zeta_s - \zeta_t = \alpha_s - \alpha_t$ $(s,t=1,\cdots,T)$ in the limit

$$N_0 \to \infty, \quad \frac{N_0}{g_{PW}^2} = \text{fixed} \sim \frac{1}{g_{R\times S^2}^2}. \tag{3.9}$$

In section 5, we will prove the prediction 1).

Next, let us discuss the prediction 2). In the gravity dual of $\mathrm{SYM}_{R\times S^2}$, we consider a configuration of disks which is periodic in the $\eta$ direction with period $\frac{\pi}{2}k$ and extract a single period. This procedure should yield the gravity dual of a theory around a vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$. In the procedure, $W=\tilde{W}$, so that the coupling constant of the resultant theory around the vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ is given by a relation

$$g_{R\times S^3/Z_k}^2 = \frac{4\pi}{k\mu}g_{R\times S^2}^2. \tag{3.10}$$

In particular, figure 5 shows the case in which the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ with the gauge group $U(N)$ is obtained. The corresponding vacuum configuration of $\mathrm{SYM}_{R\times S^2}$ is

$$\hat{\Phi} = \frac{\mu}{2}(\cdots, \underbrace{k(s-1),\cdots,k(s-1)}_{N}, \underbrace{ks,\cdots,ks}_{N}, \underbrace{k(s+1),\cdots,k(s+1)}_{N},\cdots),$$

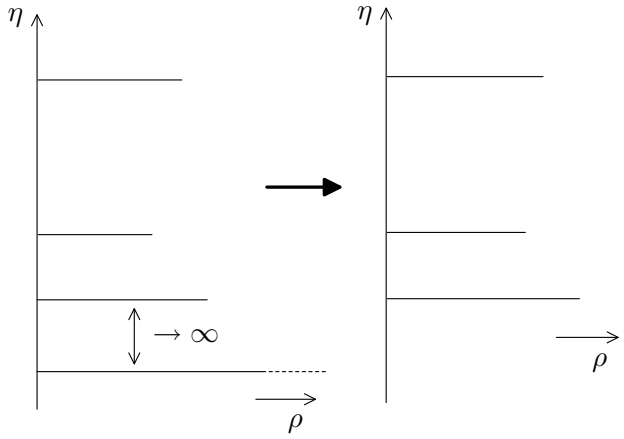


**Figure 4:** From a vacuum of the plane wave matrix model to a vacuum of $2+1$ SYM on $R \times S^2$

$$\hat{A}_1 = 0,$$
$$\hat{A}_2 = \begin{cases} \tan\frac{\theta}{2}\,\hat{\Phi} & \text{in region I} \\ -\cot\frac{\theta}{2}\,\hat{\Phi} & \text{in region II} \end{cases} \tag{3.11}$$

where $s$ runs from $-\infty$ to $\infty$. In section 6, we will show that the theory around the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ with the gauge group $U(N)$ is obtained by the theory around the vacuum labeled by (3.11) through the following procedure: we impose a condition which corresponds to the periodicity on the gravity side and extract a single period, and input the relation (3.10). This is a proof of the prediction 2) for the trivial vacuum of $\mathrm{SYN}_{R\times S^3/Z_k}$.

## 4. Spherical harmonics

In this section, we consider various spherical harmonics: the spherical harmonics on $S^3$ in section 4.1, the monopole harmonics in section 4.2, and the fuzzy sphere harmonics in section 4.3. We reveal relationship between the spherical harmonics on $S^3$ and the monopole harmonics in section 4.2, and relationship between the monopole harmonics and the fuzzy sphere harmonics in section 4.3. The latter implies that the fuzzy sphere harmonics can be regarded as a matrix regularization of the monopole harmonics. In this section, we frequently use the formula for the representations of $SU(2)$ gathered in appendix D.

### 4.1 Spherical harmonics on $S^3$

In our previous publication [31], we summarized the properties of the spherical harmonics based on [35 – 37] and found some new formula. In this subsection, we recall the properties of the spherical harmonics on $S^3$ based on [31] and add some new formula. We view $S^3$ as $G/H = SO(4)/SO(3)$, where $G = SO(4) = SU(2) \times \tilde{SU}(2)$, and the subgroup $H = SO(3)$ is naturally identified with the local 'Lorentz' group $SO(3)$ on $S^3$. We denote the generators

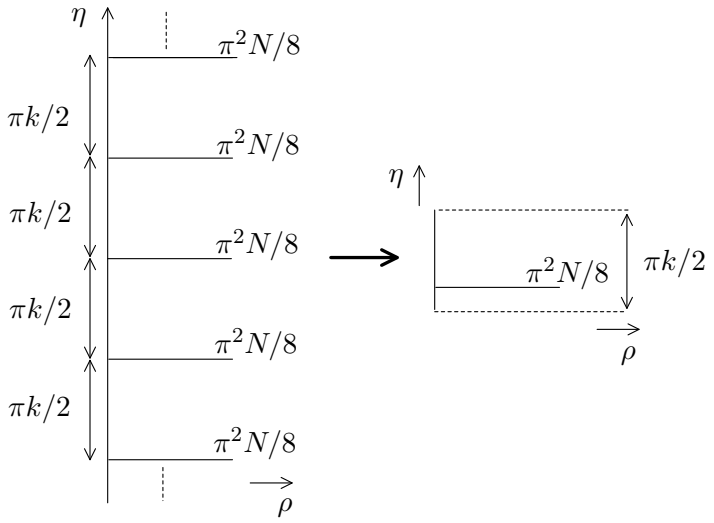


**Figure 5:** From a vacuum of $2+1$ SYM on $R \times S^2$ to the trivial vacuum of $\mathcal{N}=4$ SYM on $R \times S^3/Z_k$

of the $SU(2)$ in $G$ by $J_i$ and those of the $\tilde{SU}(2)$ in $G$ by $\tilde{J}_i$, where $i = 1, 2, 3$. Then, the generators of $H$ are represented by $S_i = J_i + \tilde{J}_i$.

The irreducible representations of $G$ are labeled by two spins, $J$ and $\tilde{J}$, which specify the irreducible representations of the $SU(2)$ and the $\tilde{SU}(2)$, respectively. We denote the basis of the $(J, \tilde{J})$ representation by $|Jm\rangle|\tilde{J}\tilde{m}\rangle$. The basis of the spin $S$ representation of $H$ is constructed in terms of $|Jm\rangle|\tilde{J}\tilde{m}\rangle$:

$$|Sn; J\tilde{J}\rangle\rangle = \sum_{m\tilde{m}} C^{Sn}_{Jm\ \tilde{J}\tilde{m}} |Jm\rangle|\tilde{J}\tilde{m}\rangle, \tag{4.1}$$

where $C^{Sn}_{Jm\ \tilde{J}\tilde{m}}$ is the Clebsch-Gordan coefficient of $SU(2)$ and the triangular inequality,

$$|J - \tilde{J}| \leq S \leq J + \tilde{J}, \tag{4.2}$$

must be satisfied.

A definite form of the representative element of $G/H$ is given by[2]

$$\Upsilon(\Omega) = e^{-i\phi J_3} e^{i\psi \tilde{J}_3} e^{-i\frac{\theta}{2}(J_1 - \tilde{J}_1)}. \tag{4.3}$$

The spin $S$ spherical harmonics on $S^3$ is given by

$$\mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}}(\Omega) = N^S_{J\tilde{J}} \langle\langle Sn; J\tilde{J}|\Upsilon^{-1}(\Omega)|Jm\rangle|\tilde{J}\tilde{m}\rangle, \tag{4.4}$$

where $N^S_{J\tilde{J}}$ is the normalization factor fixed as

$$N^S_{J\tilde{J}} = \sqrt{\frac{(2J+1)(2\tilde{J}+1)}{2S+1}}. \tag{4.5}$$

[2]We use the coordinate system given in appendix A, which is different from the one in [31].

The spherical harmonics (4.4) satisfies the orthonormal condition

$$\int \frac{d\Omega_3}{2\pi^2} \sum_n (\mathcal{Y}^{Sn}_{J_1 m_1, \tilde{J}_1 \tilde{m}_1})^* \, \mathcal{Y}^{Sn}_{J_2 m_2, \tilde{J}_2 \tilde{m}_2} = \delta_{J_1 J_2} \delta_{\tilde{J}_1 \tilde{J}_2} \delta_{m_1 m_2} \delta_{\tilde{m}_1 \tilde{m}_2}. \tag{4.6}$$

The complex conjugate of $\mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}}$ is given by

$$(\mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}})^* = (-1)^{-J+\tilde{J}-S+m-\tilde{m}+n} \, \mathcal{Y}^{S\,-n}_{J\,-m,\tilde{J}\,-\tilde{m}}. \tag{4.7}$$

The covariant derivative is understood as an algebraic manipulation:

$$\nabla_i \, \mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}}(\Omega) = -iN^S_{J\tilde{J}} \langle\langle Sn; J\tilde{J}|(J_i - \tilde{J}_i)\Upsilon^{-1}(\Omega)|Jm\rangle|\tilde{J}\tilde{m}\rangle. \tag{4.8}$$

Using this relation, it is easy to obtain the eigenvalue of the laplacian for the spin $S$ spherical harmonics:

$$\nabla^2 \mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}} = -(2J(J+1) + 2\tilde{J}(\tilde{J}+1) - S(S+1)) \, \mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}}. \tag{4.9}$$

Moreover, using (4.8) and (D.5), we find a new formula

$$\begin{aligned} C^{1r}_{S'n'\;Sn} \check{\nabla}_r \mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}} = -i(-1)^{J+\tilde{J}+S+S'-n'} \Bigg( & \sqrt{3J(J+1)(2J+1)} \begin{Bmatrix} S & S' & 1 \\ J & J & \tilde{J} \end{Bmatrix} \\ & -(-1)^{S-S'} \sqrt{3\tilde{J}(\tilde{J}+1)(2\tilde{J}+1)} \begin{Bmatrix} S & S' & 1 \\ \tilde{J} & \tilde{J} & J \end{Bmatrix} \Bigg) \tilde{Y}^{S'-n}_{Jm,\tilde{J}\tilde{m}}, \end{aligned} \tag{4.10}$$

where

$$\check{\nabla}_\pm = \mp \frac{1}{\sqrt{2}} (\nabla_1 \pm i\nabla_2), \quad \check{\nabla}_0 = \nabla_3. \tag{4.11}$$

In particular, when $S = S'$, this formula reduces to

$$C^{1r}_{Sn'\;Sn} \check{\nabla}_r \mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}} = i(-1)^{S-n'} \sqrt{3} (J(J+1) - \tilde{J}(\tilde{J}+1)) \tilde{Y}^{S-n}_{Jm,\tilde{J}\tilde{m}}. \tag{4.12}$$

By using (D.2) and (D.7), we rewrite (4.4) to an expression, in which the connection to the monopole harmonics defined in the next subsection is clear:

$$\mathcal{Y}^{Sn}_{Jm,\tilde{J}\tilde{m}} = \mathcal{K}_{Snn'} C^{Jm}_{\tilde{J}p\;Sn'} Y_{\tilde{J}p\tilde{m}}, \tag{4.13}$$

where

$$\mathcal{K}_{Snn'} = \langle Sn| e^{i\frac{\theta}{2}S_1} e^{i\phi S_3} |Sn'\rangle, \tag{4.14}$$

and $Y_{\tilde{J}p\tilde{m}} = \mathcal{Y}^{00}_{\tilde{J}p,\tilde{J}\tilde{m}}$, which is the scalar spherical harmonics. In [31], we found the compact formula for the integral of the product of three spherical harmonics,

$$\int \frac{d\Omega_3}{2\pi^2} \sum_{n_1 n_2 n_3} (\mathcal{Y}^{S_1 n_1}_{J_1 m_1, \tilde{J}_1 \tilde{m}_1})^* \, \mathcal{Y}^{S_2 n_2}_{J_2 m_2, \tilde{J}_2 \tilde{m}_2} \, \mathcal{Y}^{S_3 n_3}_{J_3 m_3, \tilde{J}_3 \tilde{m}_3} \, C^{S_1 n_1}_{S_2 n_2 \; S_3 n_3}$$

$$= \sqrt{(2S_1+1)(2J_2+1)(2\tilde{J}_2+1)(2J_3+1)(2\tilde{J}_3+1)} \begin{Bmatrix} J_1 & \tilde{J}_1 & S_1 \\ J_2 & \tilde{J}_2 & S_2 \\ J_3 & \tilde{J}_3 & S_3 \end{Bmatrix} C^{J_1 m_1}_{J_2 m_2\ J_3 m_3} C^{\tilde{J}_1 \tilde{m}_1}_{\tilde{J}_2 \tilde{m}_2\ \tilde{J}_3 \tilde{m}_3}. \tag{4.15}$$

Here we rederive the formula in a different way, starting with a particular case of the formula,

$$\int \frac{d\Omega_3}{2\pi^2} (Y_{J_1 m_1 \tilde{m}_1})^* Y_{J_2 m_2 \tilde{m}_2} Y_{J_3 m_3 \tilde{m}_3} = \sqrt{\frac{(2J_2+1)(2J_3+1)}{2J_1+1}} C^{J_1 m_1}_{J_2 m_2\ J_3 m_3} C^{J_1 \tilde{m}_1}_{J_2 \tilde{m}_2\ J_3 \tilde{m}_3}. \tag{4.16}$$

By noting

$$\sum_{n_1 n_2 n_3} C^{S_1 n_1}_{S_2 n_2\ S_3 n_3} (\mathcal{K}_{S_1 n_1 n_1'})^* \mathcal{K}_{S_2 n_2 n_2'} \mathcal{K}_{S_3 n_3 n_3'} = C^{S_1 n_1'}_{S_2 n_2'\ S_3 n_3'}, \tag{4.17}$$

we find that the lefthand side of (4.15) is equal to

$$C^{S_1 n_1}_{S_2 n_2\ S_3 n_3} C^{J_1 m_1}_{\tilde{J}_1 p_1\ S_1 n_1} C^{J_2 m_2}_{\tilde{J}_2 p_2\ S_2 n_2} C^{J_3 m_3}_{\tilde{J}_3 p_3\ S_3 n_3} \int \frac{d\Omega_3}{2\pi^2} (Y_{\tilde{J}_1 p_1 \tilde{m}_1})^* Y_{\tilde{J}_2 p_2 \tilde{m}_2} Y_{\tilde{J}_3 p_3 \tilde{m}_3}. \tag{4.18}$$

Applying (4.16) and (D.6) to this expression leads to (4.15).

As an application of the above results, we consider scalars, vectors and spinors on $S^3$. The scalar corresponds to $S = 0$. From the triangular inequality (4.2), we see that $(J, \tilde{J}) = (J, J)$. We introduce a notation for the scalar:

$$Y_{Jm\tilde{m}} \equiv \mathcal{Y}^{S=0,n=0}_{Jm,J\tilde{m}}. \tag{4.19}$$

The vector corresponds to $S = 1$. Then, the triangular inequality implies that $(J, \tilde{J})$ takes $(J+1, J)$ or $(J, J+1)$ or $(J, J)$. We assign $\rho = 1$, $\rho = -1$ and $\rho = 0$ to these three cases, respectively. We make a change of basis from the basis of the $S_3$ eigenstates to the vector basis:

$$\begin{aligned} \mathcal{Y}^1_{Jm,\tilde{J}\tilde{m}} &= \frac{1}{\sqrt{2}}(-\mathcal{Y}^{11}_{Jm,\tilde{J}\tilde{m}} + \mathcal{Y}^{1-1}_{Jm,\tilde{J}\tilde{m}}), \\ \mathcal{Y}^2_{Jm,\tilde{J}\tilde{m}} &= -\frac{i}{\sqrt{2}}(\mathcal{Y}^{11}_{Jm,\tilde{J}\tilde{m}} + \mathcal{Y}^{1-1}_{Jm,\tilde{J}\tilde{m}}), \\ \mathcal{Y}^3_{Jm,\tilde{J}\tilde{m}} &= \mathcal{Y}^{10}_{Jm,\tilde{J}\tilde{m}}. \end{aligned} \tag{4.20}$$

We introduce a notation for the vector:

$$Y^{\rho=1}_{Jm\tilde{m}i} = i\mathcal{Y}^i_{J+1\,m,J\tilde{m}}, \quad Y^{\rho=-1}_{Jm\tilde{m}i} = -i\mathcal{Y}^i_{Jm,J+1\,\tilde{m}}, \quad Y^{\rho=0}_{Jm\tilde{m}i} = \mathcal{Y}^i_{Jm,J\tilde{m}}. \tag{4.21}$$

Here the factors $\pm i$ on the right-hand side are just a convention. Note that $Y^0_{J=0\,M=(0,0)i} = 0$. The spinor corresponds to $S = \frac{1}{2}$. The triangular inequality implies that $(J, \tilde{J})$ takes $(J+\frac{1}{2}, J)$ or $(J, J+\frac{1}{2})$. We assign $\kappa = 1$ to the former and $\kappa = -1$ to the latter. We introduce a notation for the spinor:

$$Y^{\kappa=1}_{Jm\tilde{m}\alpha} = \mathcal{Y}^{S=\frac{1}{2},\alpha}_{J+\frac{1}{2}\,m,J\tilde{m}}, \quad Y^{\kappa=-1}_{Jm\tilde{m}\alpha} = \mathcal{Y}^{S=\frac{1}{2},\alpha}_{Jm,J+\frac{1}{2}\,\tilde{m}}, \tag{4.22}$$

where $\alpha$ takes $\frac{1}{2}$ and $-\frac{1}{2}$. The orthonormality condition (4.6) is translated to the scalar, the vector and the spinor as

$$\begin{aligned}
&\int \frac{d\Omega_3}{2\pi^2}\,(Y_{J_1 m_1 \tilde{m}_1})^* Y_{J_2 m_2 \tilde{m}_2} = \delta_{J_1 J_2}\delta_{m_1 m 2}\delta_{\tilde{m}_1 \tilde{m}_2},\\
&\int \frac{d\Omega_3}{2\pi^2}\,(Y^{\rho_1}_{J_1 m_1 \tilde{m}_1 i})^* Y^{\rho_2}_{J_2 m_2 \tilde{m}_2 i} = \delta_{\rho_1\rho_2}\delta_{J_1 J_2}\delta_{m_1 m_2}\delta_{\tilde{m}_1 \tilde{m}_2},\\
&\int \frac{d\Omega_3}{2\pi^2}\,(Y^{\kappa_1}_{J_1 m_1 \tilde{m}_1 \alpha})^* Y^{\kappa_2}_{J_2 m_2 \tilde{m}_2 \alpha} = \delta_{\kappa_1\kappa_2}\delta_{J_1 J_2}\delta_{m_1 m_2}\delta_{\tilde{m}_1 \tilde{m}_2},
\end{aligned} \tag{4.23}$$

while their complex conjugates are read off from (4.7) as

$$\begin{aligned}
&(Y_{Jm\tilde{m}})^* = (-1)^{m-\tilde{m}} Y_{J-m-\tilde{m}},\\
&(Y^{\rho}_{Jm\tilde{m}i})^* = (-1)^{m-\tilde{m}+1} Y^{\rho}_{J-m-\tilde{m}i},\\
&(Y^{\kappa}_{Jm\tilde{m}\alpha})^* = (-1)^{m-\tilde{m}+\kappa\alpha+1} Y^{\kappa}_{J-m-\tilde{m}-\alpha}.
\end{aligned} \tag{4.24}$$

The eigenvalues of the laplacian can be read off from (4.9):

$$\begin{aligned}
&\nabla^2\, Y_{Jm\tilde{m}} = -4J(J+1)\, Y_{Jm\tilde{m}},\\
&\nabla^2\, Y^{\pm 1}_{Jm\tilde{m}i} = -(4J(J+2)+2)\, Y^{\pm 1}_{Jm\tilde{m}i},\\
&\nabla^2\, Y^{0}_{Jm\tilde{m}i} = -(4J(J+1)-2)\, Y^{0}_{Jm\tilde{m}i},\\
&\nabla^2\, Y^{\kappa}_{Jm\tilde{m}\alpha} = -(2J(2J+3)+\frac{3}{4})\, Y^{\kappa}_{Jm\tilde{m}\alpha}.
\end{aligned} \tag{4.25}$$

Using (4.10) yields identities,

$$\begin{aligned}
&\nabla_i\, Y_{Jm\tilde{m}} = -2i\sqrt{J(J+1)}\, Y^0_{Jm\tilde{m}i},\\
&\nabla_i\, Y^{\rho}_{Jm\tilde{m}i} = -2i\delta_{\rho 0}\sqrt{J(J+1)} Y_{Jm\tilde{m}},\\
&\epsilon_{ijk}\,\nabla_j\, Y^{\rho}_{Jm\tilde{m}k} = -2\rho(J+1)\, Y^{\rho}_{Jm\tilde{m}i},\\
&\sigma^i_{\alpha\beta}\,\nabla_i\, Y^{\kappa}_{Jm\tilde{m}\beta} = -i\kappa(2J+\frac{3}{2})\, Y^{\kappa}_{Jm\tilde{m}\alpha}.
\end{aligned} \tag{4.26}$$

In [31], we defined various integrals of the product of three scalar or spinor or vector harmonics, which we call vertex coefficients:

$$\begin{aligned}
&\mathcal{C}^{J_1 m_1 \tilde{m}_1}_{J_2 m_2 \tilde{m}_2\ J_3 m_3 \tilde{m}_3} \equiv \int \frac{d\Omega_3}{2\pi^2}\,(Y_{J_1 m_1 \tilde{m}_1})^* Y_{J_2 m_2 \tilde{m}_2} Y_{J_3 m_3 \tilde{m}_3}.\\
&\mathcal{D}^{Jm\tilde{m}}_{J_1 m_1 \tilde{m}_1 \rho_1\ J_2 m_2 \tilde{m}_2 \rho_2} \equiv \int \frac{d\Omega_3}{2\pi^2}\,(Y_{Jm\tilde{m}})^* Y^{\rho_1}_{J_1 m_1 \tilde{m}_1 i} Y^{\rho_2}_{J_2 m_2 \tilde{m}_2 i}.\\
&\mathcal{E}_{J_1 m_1 \tilde{m}_1 \rho_1\ J_2 m_2 \tilde{m}_2 \rho_2\ J_3 m_3 \tilde{m}_3 \rho_3} \equiv \int \frac{d\Omega_3}{2\pi^2}\,\epsilon_{ijk}\, Y^{\rho_1}_{J_1 m_1 \tilde{m}_1 i} Y^{\rho_2}_{J_2 m_2 \tilde{m}_2 j} Y^{\rho_3}_{J_3 m_3 \tilde{m}_3 k}.\\
&\mathcal{F}^{J_1 m_1 \tilde{m}_1 \kappa_1}_{J_2 m_2 \tilde{m}_2 \kappa_2\ Jm\tilde{m}} \equiv \int \frac{d\Omega_3}{2\pi^2}\,(Y^{\kappa_1}_{J_1 m_1 \tilde{m}_1 \alpha})^* Y^{\kappa_2}_{J_2 m_2 \tilde{m}_2 \alpha} Y_{Jm\tilde{m}}.\\
&\mathcal{G}^{J_1 m_1 \tilde{m}_1 \kappa_1}_{J_2 m_2 \tilde{m}_2 \kappa_2\ Jm\tilde{m}\rho} \equiv \int \frac{d\Omega_3}{2\pi^2}\,(Y^{\kappa_1}_{J_1 m_1 \tilde{m}_1 \alpha})^* \sigma^i_{\alpha\beta} Y^{\kappa_2}_{J_2 m_2 \tilde{m}_2 \beta} Y^{\rho}_{Jm\tilde{m}i}.
\end{aligned} \tag{4.27}$$

The expressions for the vertex coefficients are obtained by using the formula (4.15) and given in appendix E.

### 4.2 Monopole harmonics

The angular momentum operator in the presence of a monopole with the magnetic charge $q$ at the origin takes the form

$$\vec{L}^{(q)} = \vec{x} \times (-i\vec{\partial} - \vec{A}^{(q)}) - q\vec{e}_r, \tag{4.28}$$

where

$$\vec{A}^{(q)} = \begin{cases} \frac{q}{r}\tan\frac{\theta}{2}\vec{e}_\phi & \text{in region I} \\ -\frac{q}{r}\cot\frac{\theta}{2}\vec{e}_\phi & \text{in region II} \end{cases} \tag{4.29}$$

The regions I and II are defined in section 2.2 and $q$ can take $0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots$ due to Dirac's quantization condition, as explained in 2.2. Noting $\vec{x} = r\vec{e}_r$, it is easy to see that neither $r$ nor the $r$-derivative appear in $\vec{L}^{(q)}$ in the polar coordinates system. Note that $\vec{L}^{(0)}$ is nothing but $\vec{L}^{(0)}$ in (2.19). $\vec{L}^{(q)}$ satisfies the $SU(2)$ algebra:

$$[L_i^{(q)}, L_j^{(q)}] = i\epsilon_{ijk}L_k^{(q)}. \tag{4.30}$$

The monopole harmonic function (section), $Y_{q,J,m}(\theta,\phi)$, was constructed by Wu and Yang [9], where $J$ takes $|q|, |q|+1, |q|+2, \cdots$ and $m$ takes $-J, -J+1, \cdots, J-1, J$. The explicit expressions for $Y_{q,J,m}$ in the regions I and II are given in [9]. It is convenient for us to multiply a phase and normalization factor:

$$\tilde{Y}_{Jmq} = (-1)^J\sqrt{4\pi}Y_{q,J,m} \tag{4.31}$$

We see from [9, 11] that $\tilde{Y}_{Jmq}$ has the following properties.

$$\begin{aligned}
&L_\pm^{(q)}\tilde{Y}_{Jmq} = \sqrt{(J\mp m)(J\pm m+1)}\tilde{Y}_{Jm\pm 1q}, \\
&L_3^{(q)}\tilde{Y}_{Jmq} = m\tilde{Y}_{Jmq}, \\
&\vec{L}^{(q)2}\tilde{Y}_{Jmq} = J(J+1)\tilde{Y}_{Jmq}, \\
&\int\frac{d\Omega_2}{4\pi}(\tilde{Y}_{Jmq})^*\tilde{Y}_{J'm'q} = \delta_{JJ'}\delta_{mm'}, \\
&(\tilde{Y}_{Jmq})^* = (-1)^{m-q}\tilde{Y}_{J-m-q}, \\
&\int\frac{d\Omega_2}{4\pi}(\tilde{Y}_{J_1m_1q_1})^*\tilde{Y}_{J_2m_2q_2}\tilde{Y}_{J_3m_3q_3} = \mathcal{C}_{J_2m_2q_2\ J_3m_3q_3}^{J_1m_1q_1} \quad \text{for } q_1 = q_2 + q_3,
\end{aligned} \tag{4.32}$$

where $\mathcal{C}_{J_2m_2q_2\ J_3m_3q_3}^{J_1m_1q_1}$ is the same as the vertex coefficient defined in (4.27). We emphasize that $J = |q|, |q|+1, |q|+2, \cdots$ and $q = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots$.

The spin $S$ monopole harmonics is defined by

$$\tilde{\mathcal{Y}}_{Jm,\tilde{J}q}^{Sn} = C_{\tilde{J}p\ Sn}^{Jm}\tilde{Y}_{\tilde{J}pq}. \tag{4.33}$$

$\tilde{\mathcal{Y}}_{Jm,\tilde{J}q}^{Sn}$ possesses the properties similar to the ones which $\mathcal{Y}_{Jm,\tilde{J}\tilde{m}}^{Sn}$ possesses with the identification $q = \tilde{m}$. The counterparts of (4.6) and (4.7) are

$$\int\frac{d\Omega_2}{4\pi}\sum_n(\tilde{\mathcal{Y}}_{J_1m_1,\tilde{J}_1q}^{Sn})^*\,\tilde{\mathcal{Y}}_{J_2m_2,\tilde{J}_2q}^{Sn} = \delta_{J_1J_2}\delta_{\tilde{J}_1\tilde{J}_2}\delta_{m_1m_2},$$

$$(\tilde{\mathcal{Y}}^{Sn}_{Jm,\tilde{J}q})^* = (-1)^{-J+\tilde{J}-S+m-q+n}\,\tilde{\mathcal{Y}}^{S\,-n}_{J\,-m,\tilde{J}\,-q}. \tag{4.34}$$

The counterpart of (4.10) is

$$C^{1r}_{S'n'\;Sn}\check{L}^{(q)}_r\tilde{\mathcal{Y}}^{Sn}_{Jm,\tilde{J}\tilde{m}} = (-1)^{-J-\tilde{J}+2S+n'+1}\sqrt{3\tilde{J}(\tilde{J}+1)(2\tilde{J}+1)}\begin{Bmatrix} S & S' & 1 \\ \tilde{J} & \tilde{J} & J \end{Bmatrix}\tilde{\mathcal{Y}}^{S'-n}_{Jm,\tilde{J}\tilde{m}}, \tag{4.35}$$

where $\check{L}^{(q)}_{\pm} = \mp\frac{1}{\sqrt{2}}(L^{(q)}_1 \pm iL^{(q)}_2)$, $\quad \check{L}^{(q)}_0 = L^{(q)}_3$. By comparing (4.13) and (4.33) and using the last identity in (4.32), we can prove the counterpart of (4.15) in the same way:

$$\begin{aligned}&\int\frac{d\Omega_2}{4\pi}\sum_{n_1n_2n_3}(\tilde{\mathcal{Y}}^{S_1n_1}_{J_1m_1,\tilde{J}_1q_1})^*\,\tilde{\mathcal{Y}}^{S_2n_2}_{J_2m_2,\tilde{J}_2q_2}\,\tilde{\mathcal{Y}}^{S_3n_3}_{J_3m_3,\tilde{J}_3q_3}\,C^{S_1n_1}_{S_2n_2\;S_3n_3}\\ &= \sqrt{(2S_1+1)(2J_2+1)(2\tilde{J}_2+1)(2J_3+1)(2\tilde{J}_3+1)}\begin{Bmatrix} J_1 & \tilde{J}_1 & S_1 \\ J_2 & \tilde{J}_2 & S_2 \\ J_3 & \tilde{J}_3 & S_3 \end{Bmatrix}C^{J_1m_1}_{J_2m_2\;J_3m_3}C^{\tilde{J}_1q_1}_{\tilde{J}_2q_2\;\tilde{J}_3q_3},\end{aligned} \tag{4.36}$$

where $q_1$ must be equal to $q_2 + q_3$.

Here we make a remark. The similarity between the spherical harmonics on $S^3$ and the monopole harmonics seen above can be understood through (4.13), (4.33) and the following equalities:

$$\begin{aligned}Y_{Jm\tilde{m}} &= (-1)^{J-m}\,\sqrt{2J+1}\,d^{(J)}_{-m,\,\tilde{m}}(\theta)e^{-i\tilde{m}(\psi-\pi/2)}e^{im(\phi+\pi/2)},\\ \tilde{Y}_{Jmq} &= \begin{cases}(-1)^J\,\sqrt{2J+1}\,d^{(J)}_{-m,\,q}(\theta)e^{i(q+m)\phi} & \text{in region I}\\ (-1)^J\,\sqrt{2J+1}\,d^{(J)}_{-m,\,q}(\theta)e^{i(-q+m)\phi} & \text{in region II}\end{cases},\end{aligned} \tag{4.37}$$

where

$$d^{(J)}_{m,\,\tilde{m}}(\theta) \equiv \langle Jm|\,e^{i\theta J_2}\,|J\tilde{m}\rangle. \tag{4.38}$$

The monopole scalar harmonics, the monopole vector harmonics and the monopole spinor harmonics are defined similarly:

$$\begin{aligned}&\tilde{Y}_{Jmq} = \tilde{\mathcal{Y}}^{00}_{Jm,\tilde{J}q},\\ &\tilde{Y}^{\rho=1}_{Jmqi} = i\tilde{\mathcal{Y}}^{i}_{J+1\,m,Jq},\quad \tilde{Y}^{\rho=-1}_{Jmqi} = -i\tilde{\mathcal{Y}}^{i}_{Jm,J+1\,q},\quad \tilde{Y}^{\rho=0}_{Jmqi} = \tilde{\mathcal{Y}}^{i}_{Jm,Jq},\\ &\tilde{Y}^{\kappa=1}_{Jmq\alpha} = \tilde{\mathcal{Y}}^{S=\frac{1}{2},\alpha}_{J+\frac{1}{2}\,m,Jq},\quad \tilde{Y}^{\kappa=-1}_{Jmq\alpha} = \tilde{\mathcal{Y}}^{S=\frac{1}{2},\alpha}_{Jm,J+\frac{1}{2}\,q},\end{aligned} \tag{4.39}$$

where $\tilde{\mathcal{Y}}^{i}_{Jm,\tilde{J}q}$ is an analogue of $\mathcal{Y}^{i}_{Jm,\tilde{J}\tilde{m}}$ and defined in terms of $\tilde{\mathcal{Y}}^{1n}_{Jm,\tilde{J}q}$'s as in (4.20). These harmonics are also orthonormal:

$$\begin{aligned}&\int\frac{d\Omega_2}{4\pi}\,(\tilde{Y}_{J_1m_1q})^*\tilde{Y}_{J_2m_2q} = \delta_{J_1J_2}\delta_{m_1m_2},\\ &\int\frac{d\Omega_2}{4\pi}\,(\tilde{Y}^{\rho_1}_{J_1m_1qi})^*\tilde{Y}^{\rho_2}_{J_2m_2qi} = \delta_{\rho_1\rho_2}\delta_{J_1J_2}\delta_{m_1m_2},\end{aligned}$$

$$\int \frac{d\Omega_2}{4\pi} (\tilde{Y}^{\kappa_1}_{J_1 m_1 q\alpha})^* \tilde{Y}^{\kappa_2}_{J_2 m_2 q\alpha} = \delta_{\kappa_1\kappa_2}\delta_{J_1 J_2}\delta_{m_1 m_2}. \tag{4.40}$$

Their complex conjugates are analogous to those of the spherical harmonics on $S^3$:

$$\begin{aligned}
&(\tilde{Y}_{Jmq})^* = (-1)^{m-q}\tilde{Y}_{J-m-q}, \quad (\tilde{Y}^{\rho}_{Jmqi})^* = (-1)^{m-q+1}\tilde{Y}^{\rho}_{J-m-qi}, \\
&(\tilde{Y}^{\kappa}_{Jmq\alpha})^* = (-1)^{m-q+\kappa\alpha+1}\tilde{Y}^{\kappa}_{J-m-q-\alpha}.
\end{aligned} \tag{4.41}$$

Using the formula (4.35) yields the identities analogous to (4.26):

$$\begin{aligned}
&\vec{L}^{(q)}\tilde{Y}_{Jmq} = \sqrt{J(J+1)}\vec{\tilde{Y}}^0_{Jmq}, \\
&\vec{L}^{(q)} \cdot \vec{\tilde{Y}}^{\rho}_{Jmq} = \sqrt{J(J+1)}\delta_{\rho 0}\tilde{Y}_{Jmq}, \\
&i\vec{L}^{(q)} \times \vec{\tilde{Y}}^{\rho}_{Jmq} + \vec{\tilde{Y}}^{\rho}_{Jmq} = \rho(J+1)\vec{\tilde{Y}}^{\rho}_{Jmq}, \\
&\left(\vec{\sigma}\cdot\vec{L}^{(q)} + \frac{3}{4}\right)\tilde{Y}^{\kappa}_{Jmq} = \kappa(J+\frac{3}{4})\tilde{Y}^{\kappa}_{Jmq}.
\end{aligned} \tag{4.42}$$

It follows from (4.15) and (4.36) that the integrals of various three monopole harmonics are equal to the corresponding integrals on $S^3$ (vertex coefficients) with the identification $q = \tilde{m}$. Namely, the following identies hold.

$$\begin{aligned}
&\int \frac{d\Omega_2}{4\pi} (\tilde{Y}_{J_1 m_1 q_1})^* \tilde{Y}_{J_2 m_2 q_2} \tilde{Y}_{J_3 m_3 q_3} = \mathcal{C}^{J_1 m_1 q_1}_{J_2 m_2 q_2\ J_3 m_3 q_3}. \\
&\int \frac{d\Omega_2}{4\pi} (\tilde{Y}_{Jmq})^* \tilde{Y}^{\rho_1}_{J_1 m_1 q_1 i} \tilde{Y}^{\rho_2}_{J_2 m_2 q_2 i} = \mathcal{D}^{Jmq}_{J_1 m_1 q_1 \rho_1\ J_2 m_2 q_2 \rho_2}. \\
&\int \frac{d\Omega_2}{4\pi} \epsilon_{ijk}\, \tilde{Y}^{\rho_1}_{J_1 m_1 q_1 i} \tilde{Y}^{\rho_2}_{J_2 m_2 q_2 j} \tilde{Y}^{\rho_3}_{J_3 m_3 q_3 k} = \mathcal{E}_{J_1 m_1 q_1 \rho_1\ J_2 m_2 q_2 \rho_2\ J_3 m_3 q_3 \rho_3}. \\
&\int \frac{d\Omega_2}{4\pi} (\tilde{Y}^{\kappa_1}_{J_1 m_1 q_1 \alpha})^* \tilde{Y}^{\kappa_2}_{J_2 m_2 q_2 \alpha} \tilde{Y}_{Jmq} = \mathcal{F}^{J_1 m_1 q_1 \kappa_1}_{J_2 m_2 q_2 \kappa_2\ Jmq}. \\
&\int \frac{d\Omega_2}{4\pi} (\tilde{Y}^{\kappa_1}_{J_1 m_1 q_1 \alpha})^* \sigma^i_{\alpha\beta} \tilde{Y}^{\kappa_2}_{J_2 m_2 q_2 \beta} \tilde{Y}^{\rho}_{Jmqi} = \mathcal{G}^{J_1 m_1 q_1 \kappa_1}_{J_2 m_2 q_2 \kappa_2\ Jmq\rho},
\end{aligned} \tag{4.43}$$

where the monopoles charges must be conserved as in the last equality in (4.32).

### 4.3 Fuzzy sphere harmonics

Let us consider the set of linear maps from a $(2j'+1)$-dimensional complex vector space $V_{j'}$ to a $(2j+1)$-dimensional complex vector space $V_j$, where $j$ and $j'$ are non-negative half-integers. We denote the set by $\mathcal{M}_{jj'}$. $\mathcal{M}_{jj'}$ is identified with the set of $(2j+1)\times(2j'+1)$ rectangular complex matrices and is a $((2j+1)\times(2j'+1))$-dimensional complex vector space. It is convenient for us to consider the basis of the spin $j$ and $j'$ representations of $SU(2)$ as a basis of $V_j$ and $V_{j'}$, respectively, and to construct a basis of $\mathcal{M}_{jj'}$ as

$$|jr\rangle\langle j'r'|, \quad (r = -j, -j+1, \cdots, j-1, j;\ r' = -j', -j'+1, \cdots, j'-1, j'). \tag{4.44}$$

Then, an arbitrary element of $\mathcal{M}_{jj'}$, $M$, is expressed as

$$M = \sum_{r,r'} M_{rr'}\, |jr\rangle\langle j'r'|. \tag{4.45}$$

One can define linear maps from $\mathcal{M}_{jj'}$ to $\mathcal{M}_{jj'}$ by its operation on the basis:

$$L_i \circ |jr\rangle\langle j'r'| = L_i|jr\rangle\langle j'r'| - |jr\rangle\langle j'r'|L_i, \tag{4.46}$$

where $L_i$ is a generator of $SU(2)$. The matrix element $M_{rr'}$ is transformed under these maps as

$$(L_i \circ M)_{rr'} = (L_i^{[j]})_{rp}M_{pr'} - M_{rp'}(L_i^{[j']})_{p'r'}, \tag{4.47}$$

where $L_i^{[j]}$ is the $(2j+1)\times(2j+1)$ representation matrix of the spin $j$ representation of $SU(2)$. These maps form a $((2j+1)\times(2j'+1))$-dimensional representation of $SU(2)$, which is in general reducible, because the following identity holds:

$$(L_i \circ L_j \circ -L_j \circ L_i\circ)|jr\rangle\langle j'r'| = i\epsilon_{ijk}L_k \circ |jr\rangle\langle j'r'|. \tag{4.48}$$

For later convenience, we introduce a positive integer constant, $N_0$, and reparameterize the dimensions of $V_j$ and $V_{j'}$ as

$$2j+1 = N_0 + \zeta, \quad 2j'+1 = N_0 + \zeta', \tag{4.49}$$

where $\zeta$ and $\zeta'$ are integers which are greater than $-N_0$. We will take the $N_0 \to \infty$ limit shortly. It will turn out that the fuzzy sphere harmonics defined below are identified with the monopole harmonics in this limit. We make a change of basis from the above basis to a new basis,

$$\hat{Y}_{Jm}^{(jj')} = \sqrt{N_0}\sum_{r,r'}(-1)^{-j'-r'-\frac{\zeta+\zeta'}{2}}C^{Jm}_{jr\,j'-r'}|jr\rangle\langle j'r'|, \tag{4.50}$$

where $J$ takes $|j-j'|, |j-j'|+1, \cdots, j+j'$ and $m$ takes $-J, -J+1, \cdots, J-1, J$. In other words, $J$ takes $\frac{1}{2}|\zeta-\zeta'|, \frac{1}{2}|\zeta-\zeta'|+1, \cdots, \frac{1}{2}(\zeta+\zeta')+N_0-1$. $N_0$ plays a role of an ultraviolet cut-off for the angular momentum. For a fixed $J$, $\hat{Y}_{Jm}^{(jj')}$ is the basis of the spin $J$ irreducible representation of $SU(2)$. Namely, using (D.3), one can show

$$\begin{aligned} L_\pm \circ \hat{Y}_{Jm}^{(jj')} &= \sqrt{(J\mp m)(J\pm m+1)}\hat{Y}_{Jm\pm1}^{(jj')}, \\ L_3 \circ \hat{Y}_{Jm}^{(jj')} &= m\hat{Y}_{Jm}^{(jj')}. \end{aligned} \tag{4.51}$$

These relations also imply

$$L_i \circ L_i \circ \hat{Y}_{Jm}^{(jj')} = J(J+1)\hat{Y}_{Jm}^{(jj')}. \tag{4.52}$$

$\hat{Y}_{Jm}^{(jj')}$ satisfies the orthonormality condition under the following normalized trace:

$$\frac{1}{N_0}\mathrm{tr}(\hat{Y}_{J_1m_1}^{(jj')\dagger}\hat{Y}_{J_2m_2}^{(jj')}) = \delta_{J_1J_2}\delta_{m_1m_2}, \tag{4.53}$$

where tr stands for the trace over $(2j'+1)\times(2j'+1)$ matrices. The hermitian conjugate of $\hat{Y}_{Jm}^{(jj')}$ is evaluated as

$$\hat{Y}_{Jm}^{(jj')\dagger} = (-1)^{m-(j-j')}\hat{Y}_{J-m}^{(j'j)}. \tag{4.54}$$

Using (D.5) yields

$$\frac{1}{N_0}\mathrm{tr}(\hat{Y}^{(j'j)\dagger}_{J_1m_1}\hat{Y}^{(j'j'')}_{J_2m_2}\hat{Y}^{(j''j)}_{J_3m_3}) \\ = (-1)^{J_1+2J_3-\frac{1}{2}\zeta-\frac{3}{2}\zeta'-\zeta''}\sqrt{N_0(2J_2+1)(2J_3+1)}C^{J_1m_1}_{J_2m_2\ J_3m_3}\begin{Bmatrix} J_1 & J_2 & J_3 \\ j'' & j & j' \end{Bmatrix}. \quad (4.55)$$

One can see from (D.8) that in the $N_0 \to \infty$ limit this equality reduces to

$$\frac{1}{N_0}\mathrm{tr}(\hat{Y}^{(j'j)\dagger}_{J_1m_1}\hat{Y}^{(j'j'')}_{J_2m_2}\hat{Y}^{(j''j)}_{J_3m_3}) = \sqrt{\frac{(2J_2+1)(2J_3+1)}{2J_1+1}}C^{J_1m_1}_{J_2m_2\ J_3m_3}C^{J_1(j'-j)}_{J_2(j'-j'')\ J_3(j''-j)}. \quad (4.56)$$

Comparing the relations (4.51), (4.52), (4.53), (4.54) and (4.56) with the relations in (4.32), one can see that $\hat{Y}^{(jj')}_{Jm}$ is identified with $\tilde{Y}_{Jmq}$ in the $N_0 \to \infty$ limit through the following correspondence:

$$\begin{aligned} j - j' &\leftrightarrow q \\ L_i\circ &\leftrightarrow L^{(q)}_i \\ \frac{1}{N_0}\mathrm{tr} &\leftrightarrow \int\frac{d\Omega_2}{4\pi}. \end{aligned} \quad (4.57)$$

In this limit, the lower bound of $J$ in $\hat{Y}^{(jj')}_{Jm}$, $|j-j'|$, remains finite and indeed corresponds to the monopole charge $q$ while the upper bound of $J$ goes to infinity, namely, the ultraviolet cut-off is removed.

The analogue of (4.33) is defined by

$$\hat{\mathcal{Y}}^{Sn}_{Jm,\tilde{J}(jj')} = C^{Jm}_{\tilde{J}p\ Sn}\hat{Y}^{(jj')}_{\tilde{J}p}, \quad (4.58)$$

which we call the spin $S$ fuzzy sphere harmonics. $\hat{\mathcal{Y}}^{Sn}_{Jm,\tilde{J}(jj')}$ shares all the properties except the integral of the product of three harmonics with $\tilde{\mathcal{Y}}^{Sn}_{Jm,\tilde{J}q}$ under the correspondence (4.57). In the $N_0 \to \infty$ limit, the trace of the product of three fuzzy sphere harmonics also coincides with the integral of the product of three monopole harmonics. The spin $S$ fuzzy sphere harmonics is, therefore, considered as a matrix regularization of the spin $S$ monopole harmonics. The counterparts of (4.34) are

$$\sum_n\frac{1}{N_0}\mathrm{tr}(\hat{\mathcal{Y}}^{Sn\dagger}_{J_1m_1,\tilde{J}_1(jj')}\tilde{\mathcal{Y}}^{Sn}_{J_2m_2,\tilde{J}_2(jj')}) = \delta_{J_1J_2}\delta_{\tilde{J}_1\tilde{J}_2}\delta_{m_1m_2}, \\ \hat{\mathcal{Y}}^{Sn\dagger}_{Jm,\tilde{J}(jj')} = (-1)^{-J+\tilde{J}-S+m-(j-j')+n}\,\hat{\mathcal{Y}}^{S\ -n}_{J\ -m,\tilde{J}\ (j'j)}. \quad (4.59)$$

The counterpart of (4.35) is

$$C^{1r}_{S'n'\ Sn}\check{L}_r\circ\hat{\mathcal{Y}}^{Sn}_{Jm,\tilde{J}(jj')} = (-1)^{-J-\tilde{J}+2S+n'+1}\sqrt{3\tilde{J}(\tilde{J}+1)(2\tilde{J}+1)}\begin{Bmatrix} S & S' & 1 \\ \tilde{J} & \tilde{J} & J \end{Bmatrix}\hat{\mathcal{Y}}^{S'-n}_{Jm,\tilde{J}(jj')}, \quad (4.60)$$

where $\check{L}_\pm \circ = \mp\frac{1}{\sqrt{2}}(L_1 \pm iL_2)\circ$, $\check{L}_0 \circ = L_3 \circ$. Using (4.55) and (D.6), it is easy to prove the following formula, which is the counterpart of (4.36),

$$\begin{aligned}
&\sum_{n_1n_2n_3} \frac{1}{N_0}\mathrm{tr}(\hat{\mathcal{Y}}^{S_1n_1\dagger}_{J_1m_1,\tilde{J}_1(j'j)}\hat{\mathcal{Y}}^{S_2n_2}_{J_2m_2,\tilde{J}_2(j'j'')}\hat{\mathcal{Y}}^{S_3n_3}_{J_3m_3,\tilde{J}_3(j''j)})\, C^{S_1n_1}_{S_2n_2\ S_3n_3} \\
&= (-1)^{\tilde{J}_1+2\tilde{J}_3-\frac{1}{2}\zeta-\frac{3}{2}\zeta'-\zeta''}\sqrt{N_0(2S_1+1)(2\tilde{J}_1+1)(2J_2+1)(2\tilde{J}_2+1)(2J_3+1)(2\tilde{J}_3+1)} \\
&\quad\times \begin{Bmatrix} J_1 & \tilde{J}_1 & S_1 \\ J_2 & \tilde{J}_2 & S_2 \\ J_3 & \tilde{J}_3 & S_3 \end{Bmatrix} C^{J_1m_1}_{J_2m_2\ J_3m_3} \begin{Bmatrix} J_1 & J_2 & J_3 \\ j'' & j & j' \end{Bmatrix}. \qquad (4.61)
\end{aligned}$$

One can see from (D.8) that in the $N_0 \to \infty$ limit, this formula reduces to

$$\begin{aligned}
&\sum_{n_1n_2n_3} \frac{1}{N_0}\mathrm{tr}(\hat{\mathcal{Y}}^{S_1n_1\dagger}_{J_1m_1,\tilde{J}_1(j'j)}\hat{\mathcal{Y}}^{S_2n_2}_{J_2m_2,\tilde{J}_2(j'j'')}\hat{\mathcal{Y}}^{S_3n_3}_{J_3m_3,\tilde{J}_3(j''j)})\, C^{S_1n_1}_{S_2n_2\ S_3n_3} \\
&= \sqrt{(2S_1+1)(2J_2+1)(2\tilde{J}_2+1)(2J_3+1)(2\tilde{J}_3+1)} \begin{Bmatrix} J_1 & \tilde{J}_1 & S_1 \\ J_2 & \tilde{J}_2 & S_2 \\ J_3 & \tilde{J}_3 & S_3 \end{Bmatrix} \\
&C^{J_1m_1}_{J_2m_2\ J_3m_3} C^{\tilde{J}_1j'-j}_{\tilde{J}_2j'-j''\tilde{J}_3j''-j}, \qquad (4.62)
\end{aligned}$$

which is equivalent to (4.36) with the identification $j - j' = q$, as anticipated.

The fuzzy sphere scalar harmonics, the fuzzy sphere vector harmonics and the fuzzy sphere spinor harmonics are defined similarly:

$$\begin{aligned}
&\hat{Y}_{Jm(jj')} = \hat{\mathcal{Y}}^{00}_{Jm,\tilde{J}(jj')} = \hat{Y}^{(jj')}_{Jm}, \\
&\hat{Y}^{\rho=1}_{Jm(jj')i} = i\hat{\mathcal{Y}}^{i}_{J+1\,m,J(jj')}, \quad \hat{Y}^{\rho=-1}_{Jm(jj')i} = -i\hat{\mathcal{Y}}^{i}_{Jm,J+1\,(jj')}, \quad \hat{Y}^{\rho=0}_{Jm(jj')i} = \hat{\mathcal{Y}}^{i}_{Jm,J(jj')}, \\
&\hat{Y}^{\kappa=1}_{Jm(jj')\alpha} = \hat{\mathcal{Y}}^{S=\frac{1}{2},\alpha}_{J+\frac{1}{2}\,m,J(jj')}, \quad \hat{Y}^{\kappa=-1}_{Jm(jj')\alpha} = \hat{\mathcal{Y}}^{S=\frac{1}{2},\alpha}_{Jm,J+\frac{1}{2}\,(jj')}, \qquad (4.63)
\end{aligned}$$

where $\hat{\mathcal{Y}}^{i}_{Jm,\tilde{J}(jj')}$ is an analogue of $\tilde{\mathcal{Y}}^{i}_{Jm,\tilde{J}q}$ and is expressed in terms of $\hat{\mathcal{Y}}^{1n}_{Jm,\tilde{J}(jj')}$'s. These harmonics are also orthonormal:

$$\begin{aligned}
&\frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\dagger}_{J_1m_1(jj')}\hat{Y}_{J_2m_2(jj')}) = \delta_{J_1J_2}\delta_{m_1m_2}, \\
&\frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\rho_1\dagger}_{J_1m_1(jj')i}\hat{Y}^{\rho_2}_{J_2m_2(jj')i}) = \delta_{\rho_1\rho_2}\delta_{J_1J_2}\delta_{m_1m_2}, \\
&\frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\kappa_1\dagger}_{J_1m_1(jj')\alpha}\hat{Y}^{\kappa_2}_{J_2m_2(jj')\alpha}) = \delta_{\kappa_1\kappa_2}\delta_{J_1J_2}\delta_{m_1m_2}. \qquad (4.64)
\end{aligned}$$

Their hermitian conjugates are analogous to the complex conjugates of the monopole harmonics:

$$\begin{aligned}
&\hat{Y}^{\dagger}_{Jm(jj')} = (-1)^{m-(j-j')}\hat{Y}_{J-m(j'j)}, \\
&\hat{Y}^{\rho\dagger}_{Jm(jj')i} = (-1)^{m-(j-j')+1}\hat{Y}^{\rho}_{J-m(j'j)i}, \\
&\hat{Y}^{\kappa\dagger}_{Jm(jj')\alpha} = (-1)^{m-(j-j')+\kappa\alpha+1}\hat{Y}^{\kappa}_{J-m(j'j)-\alpha}. \qquad (4.65)
\end{aligned}$$

Using the formula (4.60) yields the identities analogous to (4.26):

$$\begin{aligned}
&\vec{L} \circ \hat{Y}_{Jm(jj')} = \sqrt{J(J+1)}\vec{\hat{Y}}^0_{Jm(jj')}, \\
&\vec{L} \circ \cdot\vec{\hat{Y}}^{\rho}_{Jm(jj')} = \sqrt{J(J+1)}\delta_{\rho 0}\hat{Y}_{Jm(jj')}, \\
&i\vec{L} \circ \times\vec{\hat{Y}}^{\rho}_{Jm(jj')} + \vec{\hat{Y}}^{\rho}_{Jm(jj')} = \rho(J+1)\vec{\hat{Y}}^{\rho}_{Jm(jj')}, \\
&\left(\vec{\sigma}\cdot\vec{L} \circ + \frac{3}{4}\right)\hat{Y}^{\kappa}_{Jm(jj')} = \kappa(J+\frac{3}{4})\hat{Y}^{\kappa}_{Jm(jj')}.
\end{aligned} \tag{4.66}$$

We define the traces of various three fuzzy sphere harmonics, which are analogous to the vertex coefficients:

$$\begin{aligned}
&\hat{\mathcal{C}}^{J_1m_1(j'j)}_{J_2m_2(j'j'')\ J_3m_3(j''j)} \equiv \frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\dagger}_{J_1m_1(j'j)}\hat{Y}_{J_2m_2(j'j'')}\hat{Y}_{J_3m_3(j''j)}). \\
&\hat{\mathcal{D}}^{Jm(j'j)}_{J_1m_1(j'j'')rho_1\ J_2m_2(j''j)\rho_2} \equiv \frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\dagger}_{Jm(j'j)}\hat{Y}^{\rho_1}_{J_1m_1(j'j'')i}\hat{Y}^{\rho_2}_{J_2m_2(j''j)i}). \\
&\hat{\mathcal{E}}_{J_1m_1(jj')\rho_1\ J_2m_2(j'j'')\rho_2\ J_3m_3(j''j)\rho_3} \equiv \epsilon_{ijk}\frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\rho_1}_{J_1m_1(jj')i}\hat{Y}^{\rho_2}_{J_2m_2(j'j'')j}\hat{Y}^{\rho_3}_{J_3m_3(j''j)k}). \\
&\hat{\mathcal{F}}^{J_1m_1(j'j)\kappa_1}_{J_2m_2(j'j'')\kappa_2\ Jm(j''j)} \equiv \frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\kappa_1\dagger}_{J_1m_1(j'j)\alpha}\hat{Y}^{\kappa_2}_{J_2m_2(j'j'')\alpha}\hat{Y}_{Jm(j''j)}). \\
&\hat{\mathcal{G}}^{J_1m_1(j'j)\kappa_1}_{J_2m_2(j'j'')\kappa_2\ Jm(j''j)\rho} \equiv \frac{1}{N_0}\mathrm{tr}(\hat{Y}^{\kappa_1\dagger}_{J_1m_1(j'j)\alpha}\sigma^i_{\alpha\beta}\hat{Y}^{\kappa_2}_{J_2m_2(j'j'')\beta}\hat{Y}^{\rho}_{Jm(j''j)i}).
\end{aligned} \tag{4.67}$$

These can be evaluated using (4.61) and the explicit expression are given in appendix F. We see from (4.62) that these reduce to the corresponding quantities without the hat, namely the vertex coefficients, with the identification $j - j' = q$ in the $N_0 \to \infty$ limit.

## 5. $2+1$ SYM on $R \times S^2$ vs the plane wave matrix model

### 5.1 Embedding of $\mathrm{SYM}_{R\times S^2}$ into PWMM

In this subsection, we prove the prediction 1). Namely, we show that in the $N_0 \to 0$ limit the theory around the vacuum (2.36) in PWMM is equivalent to the one around the vacuum (2.29) with the identification

$$j_s - j_t = \frac{1}{2}(\alpha_s - \alpha_t) \tag{5.1}$$

and the relation between the coupling constants in (3.9).

We expand the action (A.16) around the background

$$\vec{\hat{Y}} = \vec{e}_r\hat{\Phi} + \vec{e}_\phi\hat{A}_1 - \vec{e}_\theta\hat{A}_2. \tag{5.2}$$

We make a substitution $\vec{Y} \to \vec{\hat{Y}} + \vec{Y}$ in (A.16). The terms including $\vec{Y}$ in (A.16) are evaluated as

$$\begin{aligned}
&(\vec{\mathcal{L}}X_{AB})^{(s,t)} \to \mu\vec{L}^{(q_{st})}X^{(s,t)}_{AB} - [\vec{Y}, X_{AB}]^{(s,t)}, \\
&\vec{\mathcal{Z}}^{(s,t)} \to \mu\vec{Y}^{(s,t)} + i\mu\vec{L}^{(q_{st})} \times \vec{Y}^{(s,t)} - i(\vec{Y}\times\vec{Y})^{(s,t)},
\end{aligned}$$

$$(D_0\vec{Y} - i\mu\vec{L}^{(0)}A_0)^{(s,t)} \to (D_0\vec{Y})^{(s,t)} - i\mu\vec{L}^{(q_{st})}A_0^{(s,t)}, \tag{5.3}$$

where the suffix $(s,t)$ stands for the $(s,t)$ block of an $\tilde{N}\times\tilde{N}$ matrix, which is an $N_s \times N_t$ rectangular matrix, and $s,t$ run from 1 to $T$. The monopole charge $q_{st}$ is given by

$$q_{st} = \frac{1}{2}(\alpha_s - \alpha_t). \tag{5.4}$$

By using (5.3), we obtain the theory around the vacuum (2.29):

$$\begin{aligned}
S_{R\times S^2} &= S^{free}_{R\times S^2} + S^{int}_{R\times S^2},\\
S^{free}_{R\times S^2} &= \frac{1}{g^2_{R\times S^2}}\int dt\frac{d\Omega_2}{\mu^2}\sum_{s,t}\mathrm{tr}\Bigg(\frac{1}{2}\partial_0 X^{AB(t,s)}\partial_0 X^{(s,t)}_{AB}\\
&\quad + \frac{\mu^2}{2}\vec{L}^{(q_{ts})}X^{AB(t,s)}\cdot\vec{L}^{(q_{st})}X^{(s,t)}_{AB} - \frac{\mu^2}{8}X^{AB(t,s)}X^{(s,t)}_{AB}\\
&\quad + \frac{1}{2}\partial_0\vec{Y}^{(t,s)}\cdot\partial_0\vec{Y}^{(s,t)} - \frac{1}{2}(i\mu\vec{L}^{(q_{ts})}\times\vec{Y}^{(t,s)} + \mu\vec{Y}^{(t,s)})\\
&\quad \cdot(i\mu\vec{L}^{(q_{st})}\times\vec{Y}^{(s,t)} + \mu\vec{Y}^{(s,t)})\\
&\quad - \frac{\mu^2}{2}\vec{L}^{(q_{ts})}A_0^{(t,s)}\cdot\vec{L}^{(q_{st})}A_0^{(s,t)} - i\mu\partial_0\vec{Y}^{(t,s)}\cdot\vec{L}^{(q_{st})}A_0^{(s,t)}\\
&\quad + i\psi_A^{\dagger(t,s)}\partial_0\psi^{A(s,t)} - \mu\psi_A^{\dagger(t,s)}\vec{\sigma}\cdot\vec{L}^{(q_{st})}\psi^{A(s,t)} - \frac{3\mu}{4}\psi_A^{\dagger(t,s)}\psi^{A(s,t)}\Bigg),\\
S^{int}_{R\times S^2} &= \frac{1}{g^2_{R\times S^2}}\int dt\frac{d\Omega_2}{\mu^2}\sum_{s,t}\mathrm{tr}\Bigg(-i\partial_0 X^{(t,s)}_{AB}[A_0,X^{AB}]^{(s,t)} - \frac{1}{2}[A_0,X_{AB}]^{(t,s)}[A_0,X^{AB}]^{(s,t)}\\
&\quad - \mu\vec{L}^{(q_{ts})}X^{(t,s)}_{AB}\cdot[\vec{Y},X^{AB}]^{(s,t)} + \frac{1}{2}[\vec{Y},X_{AB}]^{(t,s)}\cdot[\vec{Y},X^{AB}]^{(s,t)}\\
&\quad + \frac{1}{4}[X_{AB},X_{CD}]^{(t,s)}[X^{AB},X^{CD}]^{(s,t)} - \frac{1}{2}[\vec{Y},A_0]^{(t,s)}\cdot[\vec{Y},A_0]^{(s,t)}\\
&\quad - i\partial_0\vec{Y}^{(t,s)}\cdot[A_0,\vec{Y}]^{(s,t)} - \mu[A_0,\vec{Y}]^{(t,s)}\cdot\vec{L}^{(q_{st})}A_0^{(s,t)}\\
&\quad + i(i\mu\vec{L}^{(q_{ts})}\times\vec{Y}^{(t,s)} + \mu\vec{Y}^{(t,s)})\cdot(\vec{Y}\times\vec{Y})^{(s,t)}\\
&\quad + \frac{1}{2}(\vec{Y}\times\vec{Y})^{(t,s)}\cdot(\vec{Y}\times\vec{Y})^{(s,t)}\\
&\quad + \psi_A^{\dagger(t,s)}[A_0,\psi^A]^{(s,t)} + \psi_A^{\dagger(t,s)}\vec{\sigma}\cdot[\vec{Y},\psi^A]^{(s,t)}\\
&\quad - \psi^{AT(t,s)}\sigma^2[X_{AB},\psi^B]^{(s,t)} + \psi_A^{\dagger(t,s)}\sigma^2[X^{AB},\psi^*_B]^{(s,t)}\Bigg),
\end{aligned} \tag{5.5}$$

where tr should be understood as the trace over square matrices with a certain size which are the products of some rectangular matrices.

Moreover, we make the mode expansion for the fields in terms of the monopole harmonics as

$$A_0^{(s,t)} = \sum_{J\geq|q_{st}|}\sum_{m=-J}^{J} b^{(s,t)}_{Jm}\tilde{Y}_{Jmq_{st}}, \qquad X^{(s,t)}_{AB} = \sum_{J\geq|q_{st}|}\sum_{m=-J}^{J} x^{(s,t)}_{ABJm}\tilde{Y}_{Jmq_{st}},$$

$$
\begin{aligned}
\psi^{A(s,t)} &= \sum_{\kappa=\pm 1} \sum_{\tilde{U}\geq|q_{st}|} \sum_{m=-U}^{U} \psi^{A(s,t)}_{Jm\kappa} \tilde{Y}^{\kappa}_{Jmq_{st}} \\
&= \sum_{J\geq|q_{st}|} \sum_{m=-J-\frac{1}{2}}^{J+\frac{1}{2}} \psi^{A(s,t)}_{Jm1} \tilde{Y}^{1}_{Jmq_{st}} + \sum_{J\geq|q_{st}|-\frac{1}{2}} \sum_{m=-J}^{J} \psi^{A(s,t)}_{Jm-1} \tilde{Y}^{-1}_{Jmq_{st}}, \\
\vec{Y}^{(s,t)} &= \sum_{\rho=-1}^{1} \sum_{\tilde{Q}\geq|q_{st}|} \sum_{m=-Q}^{Q} y^{(s,t)}_{Jm\rho} \vec{\tilde{Y}}^{\rho}_{Jmq_{st}}, \\
&= \sum_{J\geq|q_{st}|} \sum_{m=-J-1}^{J+1} y^{(s,t)}_{Jm1} \vec{\tilde{Y}}^{1}_{Jmq_{st}} + \sum_{J\geq|q_{st}|} \sum_{m=-J}^{J} y^{(s,t)}_{Jm0} \vec{\tilde{Y}}^{0}_{Jmq_{st}} \\
&+ \sum_{J\geq|q_{st}|-1} \sum_{m=-J}^{J} y^{(s,t)}_{Jm-1} \vec{\tilde{Y}}^{-1}_{Jmq_{st}},
\end{aligned} \tag{5.6}
$$

where $U \equiv J + \frac{1+\kappa}{4}, \tilde{U} \equiv J + \frac{1-\kappa}{4}, Q \equiv J + \frac{(1+\rho)\rho}{2}$ and $\tilde{Q} \equiv J - \frac{(1-\rho)\rho}{2}$. Due to (4.41), the conditions $A_0^{(s,t)\dagger} = A_0^{(t,s)}, X_{AB}^{(s,t)\dagger} = X^{AB(t,s)}$ and $\vec{Y}^{(s,t)\dagger} = \vec{Y}^{(t,s)}$ imply

$$
\begin{aligned}
b^{(s,t)\dagger}_{Jm} &= (-1)^{m-q_{st}} b^{(t,s)}_{J-m}, \qquad x^{(s,t)\dagger}_{ABJm} = (-1)^{m-q_{st}} x^{AB(t,s)}_{J-m}, \\
y^{(s,t)\dagger}_{Jm\rho} &= (-1)^{m-q_{st}+1} y^{(t,s)}_{J-m\rho}.
\end{aligned} \tag{5.7}
$$

By substituting (5.6) into (5.5) and using (4.40), (4.42) and (4.43), we obtain the mode-expanded form of the theory:

$$
\begin{aligned}
S^{free}_{R\times S^2} = \frac{4\pi}{g^2_{R\times S^2}} \int \frac{dt}{\mu^2} \mathrm{tr} \Bigg[ & \frac{1}{2} \partial_0 x^{(s,t)\dagger}_{AB\omega} \partial_0 x^{(s,t)}_{AB\omega} - \frac{\mu^2}{2} \left(J+\frac{1}{2}\right)^2 x^{(s,t)\dagger}_{AB\omega} x^{(s,t)}_{AB\omega} \\
& + \frac{1}{2} \partial_0 y^{(s,t)\dagger}_{\omega\rho} \partial_0 y^{(s,t)}_{\omega\rho} - \frac{\mu^2}{2} \rho^2 (J+1)^2 y^{(s,t)\dagger}_{\omega\rho} y^{(s,t)}_{\omega\rho} \\
& + \frac{\mu^2}{2} J(J+1) b^{(s,t)\dagger}_{\omega} b^{(s,t)}_{\omega} - i\mu\sqrt{J(J+1)} \partial_0 y^{(s,t)\dagger}_{\omega 0} b^{(s,t)}_{\omega} \\
& + i\psi^{(s,t)\dagger}_{A\omega\kappa} \partial_0 \psi^{A(s,t)}_{\omega\kappa} - \mu\kappa \left(J+\frac{3}{4}\right) \psi^{(s,t)\dagger}_{A\omega\kappa} \psi^{A(s,t)}_{\omega\kappa} \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
S^{int}_{R\times S^2} = \frac{4\pi}{g^2_{R\times S^2}} \int \frac{dt}{\mu^2} \mathrm{tr} \Bigg[ & -i\mathcal{C}_{\omega_1 q_{st}\ \omega_2 q_{tu}\ \omega_3 q_{us}} \partial_0 x^{(s,t)}_{AB,\omega_1} \left( b^{(t,u)}_{\omega_2} x^{AB(u,s)}_{\omega_3} - x^{AB(t,u)}_{\omega_2} b^{(u,s)}_{\omega_3} \right) \\
& - \frac{1}{2} \mathcal{C}^{\omega q}_{\omega_1 q_{st}\ \omega_2 q_{tu}} \mathcal{C}_{\omega q\ \omega_3 q_{uv}\ \omega_4 q_{vs}} \left( b^{(s,t)}_{\omega_1} x^{(t,u)}_{AB,\omega_2} - x^{(s,t)}_{AB,\omega_1} b^{(t,u)}_{\omega_2} \right) \left( b^{(u,v)}_{\omega_3} x^{AB(v,s)}_{\omega_4} - x^{AB(u,v)}_{\omega_3} b^{(v,s)}_{\omega_4} \right) \\
& - \mu\sqrt{J_1(J_1+1)} \left( \mathcal{D}_{\omega_2 q_{us}\ \omega_1 q_{st} 0\ \omega q_{tu}\rho} x^{(s,t)}_{AB\omega_1} y^{(t,u)}_{\omega\rho} x^{AB(u,s)}_{\omega_2} - \mathcal{D}_{\omega q_{tu}\ \omega_1 q_{st} 0\ \omega_2 q_{us}\rho_2} x^{(s,t)}_{AB\omega_1} x^{AB(t,u)}_{\omega} y^{(u,s)}_{\omega_2\rho_2} \right) \\
& + \mathcal{C}^{\omega q}_{\omega_1 q_{st}\ \omega_3 q_{uv}} \mathcal{D}_{\omega q\ \omega_2 q_{tu}\rho_2\ \omega_4 q_{vs}\rho_4} x^{(s,t)}_{AB\omega_1} y^{(t,u)}_{\omega_2\rho_2} x^{AB(u,v)}_{\omega_3} y^{(v,s)}_{\omega_4\rho_4} \\
& - \mathcal{C}^{\omega q}_{\omega_1 q_{st}\ \omega_4 q_{vs}} \mathcal{D}_{\omega q\ \omega_2 q_{tu}\rho_2\ \omega_3 q_{uv}\rho_3} x^{(s,t)}_{AB\omega_1} y^{(t,u)}_{\omega_2\rho_2} y^{(u,v)}_{\omega_3\rho_3} x^{AB(v,s)}_{\omega_4} \\
& + \frac{1}{4} \mathcal{C}^{\omega q}_{\omega_1 q_{st}\ \omega_2 q_{tu}} \mathcal{C}_{\omega q\ \omega_3 q_{uv}\ \omega_4 q_{vs}} \left( x^{(s,t)}_{AB\omega_1} x^{(t,u)}_{CD\omega_2} - x^{(s,t)}_{CD\omega_1} x^{(t,u)}_{AB\omega_2} \right) \left( x^{AB(u,v)}_{\omega_3} x^{CD(v,s)}_{\omega_4} - x^{CD(u,v)}_{\omega_3} x^{AB(v,s)}_{\omega_4} \right) \\
& - i \Big( \mathcal{D}_{\omega q_{tu}\ \omega_1 q_{st}\rho_1\ \omega_2 q_{us}\rho_2} \partial_0 y^{(s,t)}_{\omega_1\rho_1} b^{(t,u)}_{\omega} y^{(u,s)}_{\omega_2\rho_2} - \mathcal{D}_{\omega_2 q_{us}\ \omega_1 q_{st}\rho_1\ \omega q_{tu}\rho} \partial_0 y^{(s,t)}_{\omega_1\rho_1} y^{(t,u)}_{\omega\rho} b^{(u,s)}_{\omega_2} \Big) \\
& + \mu\sqrt{J_1(J_1+1)} \left( \mathcal{D}_{\omega_2 q_{us}\ \omega_1 q_{st} 0\ \omega q_{tu}\rho} b^{(s,t)}_{\omega_1} y^{(t,u)}_{\omega\rho} b^{(u,s)}_{\omega_2} - \mathcal{D}_{\omega q_{tu}\ \omega_1 q_{st} 0\ \omega_2 q_{us}\rho_2} b^{(s,t)}_{\omega_1} b^{(t,u)}_{\omega} y^{(u,s)}_{\omega_2\rho_2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \mathcal{C}^{\omega q}_{\omega_1 q_{st}\ \omega_3 q_{uv}} \mathcal{D}_{\omega q\ \omega_2 q_{tu}\rho_2\ \omega_4 q_{vs}\rho_4} b^{(s,t)}_{\omega_1} y^{(t,u)}_{\omega_2\rho_2} b^{(u,v)}_{\omega_3} y^{(v,s)}_{\omega_4\rho_4} \\
&+ \mathcal{C}^{\omega q}_{\omega_1 q_{st}\ \omega_4 q_{vs}} \mathcal{D}_{\omega q\ \omega_2 q_{tu}\rho_2\ \omega_3 q_{uv}\rho_3} b^{(s,t)}_{\omega_1} y^{(t,u)}_{\omega_2\rho_2} y^{(u,v)}_{\omega_3\rho_3} b^{(v,s)}_{\omega_4} \\
&+ i\mu\rho_1(J_1+1)\mathcal{E}_{\omega_1 q_{st}\rho_1\ \omega_2 q_{tu}\rho_2\ \omega_3 q_{us}\rho_3} y^{(s,t)}_{\omega_1\rho_1} y^{(t,u)}_{\omega_2\rho_2} y^{(u,s)}_{\omega_3\rho_3} \\
&+ \frac{1}{2}\Big(\mathcal{D}^{\omega q}_{\omega_1 q_{st}\rho_1\ \omega_3 q_{uv}\rho_3} \mathcal{D}_{\omega q\ \omega_2 q_{tu}\rho_2\ \omega_4 q_{vs}\rho_4} - \mathcal{D}^{\omega q}_{\omega_1 q_{st}\rho_1\ \omega_4 q_{vs}\rho_4} \mathcal{D}_{\omega q\ \omega_2 q_{tu}\rho_2\ \omega_3 q_{uv}\rho_3}\Big) y^{(s,t)}_{\omega_1\rho_1} y^{(t,u)}_{\omega_2\rho_2} y^{(u,v)}_{\omega_3\rho_3} y^{(v,s)}_{\omega_4\rho_4} \\
&+ \mathcal{F}^{\omega_1 q_{st}\kappa_1}_{\omega_2 q_{ut}\kappa_2\ \omega q_{su}} \psi^{(s,t)\dagger}_{A\omega_1\kappa_1} b^{(s,u)}_{\omega} \psi^{A(u,t)}_{\omega_2\kappa_2} - \mathcal{F}^{\omega_1 q_{st}\kappa_1}_{\omega q_{su}\kappa\ \omega_2 q_{ut}} \psi^{(s,t)\dagger}_{A\omega_1\kappa_1} \psi^{A(s,u)}_{\omega\rho} b^{(u,t)}_{\omega_2} \\
&+ \mathcal{G}^{\omega_1 q_{st}\kappa_1}_{\omega_2 q_{ut}\kappa_2\ \omega q_{su}\rho} \psi^{(s,t)\dagger}_{A\omega_1\kappa_1} y^{(s,u)}_{\omega\rho} \psi^{A(u,t)}_{\omega_2\kappa_2} - \mathcal{G}^{\omega_1 q_{st}\kappa_1}_{\omega q_{su}\kappa\ \omega_2 q_{ut}\rho_2} \psi^{(s,t)\dagger}_{A\omega\kappa_1} \psi^{A(s,u)}_{\omega\kappa} y^{(u,t)}_{\omega_2\rho_2} \\
&- i(-1)^{m_1-q_{ts}+\frac{\kappa_1}{2}} \mathcal{F}^{J_1-m_1-q_{ts}\kappa_1}_{\omega_2 q_{ut}\kappa_2\ \omega q_{su}} \psi^{A(t,s)}_{J_1 m_1\kappa_1} x^{(s,u)}_{AB\omega} \psi^{B(u,t)}_{\omega_2\kappa_2} \\
&+ i(-1)^{m_1-q_{ts}+\frac{\kappa_1}{2}} \mathcal{F}^{J_1-m_1-q_{ts}\kappa_1}_{\omega q_{su}\kappa_2\ \omega_2 q_{ut}} \psi^{A(t,s)}_{J_1 m_1\kappa_1} \psi^{B(s,u)}_{\omega\kappa} x^{(u,t)}_{AB\omega_2} \\
&+ i(-1)^{m_2-q_{tu}+\frac{\kappa_2}{2}} \mathcal{F}^{\omega_1 q_{st}\kappa_1}_{J_2-m_2-q_{tu}\kappa_2\ \omega q_{su}} \psi^{(s,t)\dagger}_{A\omega_1\kappa_1} x^{AB(s,u)}_{\omega} \psi^{(t,u)\dagger}_{BJ_2 m_2\kappa_2} \\
&- i(-1)^{m-q_{us}+\frac{\kappa}{2}} \mathcal{F}^{\omega_1 q_{st}\kappa_1}_{J-m-q_{us}\kappa\ \omega_2 q_{ut}} \psi^{(s,t)\dagger}_{A\omega_1\kappa_1} \psi^{(u,s)\dagger}_{BJm\kappa} x^{AB(u,t)}_{\omega_2}\Big],
\end{aligned}
\tag{5.8}
$$

where the summation over the indices that appear twice or more than twice is assumed and we have introduced the abbreviated notations: $\omega$ represents a pair, $(J, m)$.

Similarly, we expand the action (A.17) around the vacuum (2.36). We make a substitution $\vec{Y} \to \hat{\vec{Y}} + \vec{Y}$ in (A.17), where $\hat{Y}_i = -\mu L_i$ and $L_i$ is given in (2.36). The result is

$$
\begin{aligned}
&S_{PW} = S^{free}_{PW} + S^{int}_{PW}, \\
&S^{free}_{PW} = \frac{1}{g^2_{PW}} \int \frac{dt}{\mu^2} \sum_{s,t} \mathrm{tr}\Bigg( \frac{1}{2}\partial_0 X^{AB(t,s)} \partial_0 X^{(s,t)}_{AB} + \frac{\mu^2}{2} \vec{L} \circ X^{AB(t,s)} \cdot \vec{L} \circ X^{(s,t)}_{AB} \\
&\qquad - \frac{\mu^2}{8} X^{AB(t,s)} X^{(s,t)}_{AB} + \frac{1}{2}\partial_0 \vec{Y}^{(t,s)} \cdot \partial_0 \vec{Y}^{(s,t)} \\
&\qquad - \frac{1}{2}(i\mu\vec{L} \circ \times \vec{Y}^{(t,s)} + \mu\vec{Y}^{(t,s)}) \cdot (i\mu\vec{L} \circ \times \vec{Y}^{(s,t)} + \mu\vec{Y}^{(s,t)}) \\
&\qquad - \frac{\mu^2}{2} \vec{L} \circ A^{(t,s)}_0 \cdot \vec{L} \circ A^{(s,t)}_0 - i\mu\partial_0 \vec{Y}^{(t,s)} \cdot \vec{L} \circ A^{(s,t)}_0 \\
&\qquad + i\psi^{\dagger(t,s)}_A \partial_0 \psi^{A(s,t)} - \mu\psi^{\dagger(t,s)}_A \vec{\sigma} \cdot \vec{L} \circ \psi^{A(s,t)} - \frac{3\mu}{4} \psi^{\dagger(t,s)}_A \psi^{A(s,t)} \Bigg), \\
&S^{int}_{PW} = \frac{1}{g^2_{PW}} \int \frac{dt}{\mu^2} \sum_{s,t} \mathrm{tr}\Bigg( -i\partial_0 X^{(t,s)}_{AB} [A_0, X^{AB}]^{(s,t)} - \frac{1}{2}[A_0, X_{AB}]^{(t,s)} [A_0, X^{AB}]^{(s,t)} \\
&\qquad - \mu\vec{L} \circ X^{(t,s)}_{AB} \cdot [\vec{Y}, X^{AB}]^{(s,t)} + \frac{1}{2}[\vec{Y}, X_{AB}]^{(t,s)} \cdot [\vec{Y}, X^{AB}]^{(s,t)} \\
&\qquad + \frac{1}{4}[X_{AB}, X_{CD}]^{(t,s)} [X^{AB}, X^{CD}]^{(s,t)} - \frac{1}{2}[\vec{Y}, A_0]^{(t,s)} \cdot [\vec{Y}, A_0]^{(s,t)} \\
&\qquad - i(\partial_0 \vec{Y})^{(t,s)} \cdot [A_0, \vec{Y}]^{(s,t)} - \mu[A_0, \vec{Y}]^{(t,s)} \cdot (\vec{L} \circ A_0)^{(s,t)} \\
&\qquad + i(i\mu\vec{L} \circ \times \vec{Y}^{(t,s)} + \mu\vec{Y}^{(t,s)}) \cdot (\vec{Y} \times \vec{Y})^{(s,t)} \\
&\qquad + \frac{1}{2}(\vec{Y} \times \vec{Y})^{(t,s)} \cdot (\vec{Y} \times \vec{Y})^{(s,t)} \\
&\qquad + \psi^{\dagger(t,s)}_A [A_0, \psi^A]^{(s,t)} + \psi^{\dagger(t,s)}_A \vec{\sigma} \cdot [\vec{Y}, \psi^A]^{(s,t)}
\end{aligned}
$$

$$- \psi^{AT(t,s)}\sigma^2[X_{AB},\psi^B]^{(s,t)} + \psi_A^{\dagger(t,s)}\sigma^2[X^{AB},\psi_B^*]^{(s,t)}\Bigg). \tag{5.9}$$

Here the suffix $(s,t)$ stands for the $(s,t)$ 'large' block of an $\hat{N}\times\hat{N}$ matrix, which is an $N_s(2j_s+1)\times N_t(2j_t+1)$ rectangular matrix, and $s,t$ run from 1 to $T$. The reader would notice resemblance between (5.5) and (5.9). We make a mode expansion analogous to (5.6):

$$\begin{aligned}
A_0^{(s,t)} &= \sum_{J=|j_s-j_t|}^{j_s+j_t}\sum_{m=-J}^{J} b_{Jm}^{(s,t)}\otimes\hat{Y}_{Jm(j_sj_t)}, \qquad X_{AB}^{(s,t)} = \sum_{J=|j_s-j_t|}^{j_s+j_t}\sum_{m=-J}^{J} x_{ABJm}^{(s,t)}\otimes\hat{Y}_{Jm(j_sj_t)},\\
\psi^{A(s,t)} &= \sum_{\kappa=\pm1}\sum_{\tilde{U}=|j_s-j_t|}^{j_s+j_t}\sum_{m=-U}^{U}\psi_{Jm\kappa}^{A(s,t)}\otimes\hat{Y}^{\kappa}_{Jm(j_sj_t)}\\
&= \sum_{J=|j_s-j_t|}^{j_s+j_t}\sum_{m=-J-\frac{1}{2}}^{J+\frac{1}{2}}\psi_{Jm1}^{A(s,t)}\otimes\hat{Y}^{1}_{Jm(j_sj_t)}\sum_{J=|j_s-j_t|-\frac{1}{2}}^{j_s+j_t-\frac{1}{2}}\sum_{m=-J}^{J}\psi_{Jm-1}^{A(s,t)}\otimes\hat{Y}^{-1}_{Jm(j_sj_t)},\\
\vec{Y}^{(s,t)} &= \sum_{\rho=-1}^{1}\sum_{\tilde{Q}=|j_s-j_t|}^{j_s+j_t}\sum_{m=-Q}^{Q} y_{Jm\rho}^{(s,t)}\otimes\vec{\hat{Y}}^{\rho}_{Jm(j_sj_t)}\\
&= \sum_{J=|j_s-j_t|}^{j_s+j_t}\sum_{m=-J-1}^{J+1} y_{Jm1}^{(s,t)}\otimes\vec{\hat{Y}}^{1}_{Jm(j_sj_t)} + \sum_{J=|j_s-j_t|}^{j_s+j_t}\sum_{m=-J}^{J} y_{Jm0}^{(s,t)}\otimes\vec{\hat{Y}}^{0}_{Jm(j_sj_t)}\\
&\quad + \sum_{J=|j_s-j_t|-1}^{j_s+j_t-1}\sum_{m=-J}^{J} y_{Jm-1}^{(s,t)}\otimes\vec{\hat{Y}}^{-1}_{Jm(j_sj_t)},
\end{aligned} \tag{5.10}$$

In the above expressions, the both sides are $N_s(2j_s+1)\times N_t(2j_t+1)$ matrices and the modes in the righthand sides such as $x_{ABJm}^{(s,t)}$ are $N_s\times N_t$ matrices. Due to (4.65), (5.7) also holds for this case.

By substituting (5.10) into (5.9) and using (4.64), (4.66) and (4.67), we obtain the mode-expanded form of the theory around the vacuum (2.36). By setting

$$\frac{4\pi}{g^2_{R\times S^2}} = \frac{N_0}{g^2_{PW}} \tag{5.11}$$

and

$$q_{st} = j_s - j_t, \tag{5.12}$$

it is easy to see that the free part completely coincides with $S^{free}_{R\times S^2}$ in (5.8) while the interaction part is obtained by attaching the hat to the vertex coefficients in $S^{int}_{R\times S^2}$ and replacing $q_{st}$ in the vertex coefficients with $(j_sj_t)$. As seen in section 4.3, the vertex coefficients with the hat reduce to the vertex coefficients with the identification $q=j-j'$ in the $N_0\to\infty$ limit. Thus, in the $N_0\to\infty$ limit, the interaction part also coincides with $S^{int}_{R\times S^2}$ in (5.8). Furthermore, the relation (5.12) is equivalent to (5.1), and the relation (5.11) is consistent with (3.9). Thus we have completed the proof of the prediction 1).



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### 5.2 Topologically nontrivial configurations on fuzzy spheres

In this subsection, we comment on a relation of our results in the previous subsection with the works [19, 20].

The authors of [19, 20] considered a configuration

$$Y_i = -\mu L_i = -\mu \begin{pmatrix} L_i^{[j_1]} & 0 \\ 0 & L_i^{[j_2]} \end{pmatrix} \tag{5.13}$$

as a topologically nontrivial gauge configuration, where $\zeta_1 - \zeta_2 = 2\alpha$ $(2j_1 + 1 = N_0 + \zeta_1,\ 2j_2 + 1 = N_0 + \zeta_2)$ with $\alpha$ an integer. They introduced the topological index on a fuzzy sphere which can be defined for the configuration (5.13). Their topological index for (5.13) is equal to $\frac{1}{2}|\zeta_1 - \zeta_2| = |\alpha|$, and they claimed that it coincides with the winding number $\pi_2(SU(2)/U(1))$ in the continuum limit ($N_0 \to \infty$ limit). Actually, in the case in which $\alpha = 1$, they directly obtained from (5.13) the 't Hooft-Polyakov monopole solution, which has the winding number one.

According to our result in the previous subsection, the vacuum configuration of $\mathrm{SYM}_{R\times S^2}$ corresponding to (5.13) in the $N_0 \to \infty$ limit is

$$\begin{aligned} &\hat{\Phi} = \frac{\mu}{2} \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \\ &\hat{A}_1 = 0, \\ &\hat{A}_2 = \begin{cases} \tan\frac{\theta}{2}\,\hat{\Phi} & \text{in region I} \\ -\cot\frac{\theta}{2}\,\hat{\Phi} & \text{in region II} \end{cases}, \end{aligned} \tag{5.14}$$

where we have extracted the $SU(2)$ part separating the decoupled $U(1)$ part. Namely, for generic $\alpha$, we found the gauge configuration on $S^2$ to which (5.13) reduces in the $N_0 \to \infty$ limit. In the following, we check a consistency that the configuration (5.14) has the winding number $|\alpha|$.

We define a gauge invariant quantity by

$$\begin{aligned} \mathcal{F}_{a'b'} &= \mathrm{Tr}(\tilde{\Phi} F_{a'b'} - \tilde{\Phi}[D_{a'}\tilde{\Phi}, D_{b'}\tilde{\Phi}]) \\ &= \mathrm{Tr}(\nabla_{a'}(\tilde{\Phi} A_{b'}) - \nabla_{b'}(\tilde{\Phi} A_{a'}) - \tilde{\Phi}[\nabla_{a'}\tilde{\Phi}, \nabla_{b'}\tilde{\Phi}]), \end{aligned} \tag{5.15}$$

where

$$\tilde{\Phi} = \frac{\Phi}{\sqrt{2\mathrm{Tr}\Phi^2}}. \tag{5.16}$$

Then the topological charge is given by

$$Q = \frac{1}{8\pi} \int d\theta d\phi \sin\theta \mathcal{F}_{12} \tag{5.17}$$

Actually, for configurations where $f_{a'b'} = \mathrm{Tr}(\nabla_{a'}(\tilde{\Phi} A_{b'}) - \nabla_{b'}(\tilde{\Phi} A_{a'}))$ is total derivative, (5.17) reduces to

$$Q = -\frac{1}{8\pi} \int d\theta d\phi \sin\theta \mathrm{Tr}(\tilde{\Phi}[\nabla_1\tilde{\Phi}, \nabla_2\tilde{\Phi}]), \tag{5.18}$$

which is the winding number $\pi_2(SU(2)/U(1))$. For the configuration (5.14), $f_{a'b'}$ is not total derivative while $\mathrm{Tr}(\tilde{\Phi}[\nabla_{a'}\tilde{\Phi}, \nabla_{b'}\tilde{\Phi}])$ vanishes. $Q$ is evaluated from (5.17) as $Q = |\alpha|$. One can also obtain the same value for $Q$ from (5.18) by applying a singular gauge transformation to (5.14). In the region II, it takes the form

$$V = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\alpha\phi} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2}e^{i\alpha\phi} \end{pmatrix}. \tag{5.19}$$

The resultant gauge transformed configuration is

$$\begin{aligned} \hat{\Phi} &\rightarrow V^\dagger \hat{\Phi} V = \frac{\mu\alpha}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{i\alpha\phi} \\ \sin\theta e^{-i\alpha\phi} & -\cos\theta \end{pmatrix}, \\ \hat{A}_1 &\rightarrow V^\dagger \hat{A}_1 V + iV^\dagger \nabla_1 V = \frac{i\mu}{2}\begin{pmatrix} 0 & e^{i\alpha\phi} \\ -e^{-i\alpha\phi} & 0 \end{pmatrix}, \\ \hat{A}_2 &\rightarrow V^\dagger \hat{A}_2 V + iV^\dagger \nabla_2 V = \frac{\mu\alpha}{2}\begin{pmatrix} \sin\theta & -\cos\theta e^{i\alpha\phi} \\ -\cos\theta e^{-i\alpha\phi} & -\sin\theta \end{pmatrix}. \end{aligned} \tag{5.20}$$

In the region I, the same configuration of the fields are obtained by the gauge transformation $V_{I\to II}V$, where $V_{I\to II}$ is given in (2.30). Note that the single-valuedness of $V$ and the gauge transformed fields requires $\alpha$ to be an integer. For the gauge transformed configuration (5.20), $f_{a'b'}$ vanishes and (5.18) indeed gives $Q = |\alpha|$. Thus, for the configuration (5.14) with generic $\alpha$, $|\alpha|$ is interpreted as the winding number. For $\alpha = \pm 1$, it is easy to check that (5.20) is nothing but the 't Hooft-Polyakov monopole solution, which is smooth everywhere on $S^2$. For $\alpha \neq \pm 1$, although the gauge fields in (5.20) are not smooth everywhere, $\Phi$ is smooth everywhere and $Q$ is given by (5.18).

When $\zeta_1 - \zeta_2$ in (5.13) is an odd integer, one can also consider the corresponding configuration on $S^2$ (5.14) in which $2\alpha$ is equal to the odd integer $\zeta_1 - \zeta_2$. This configuration indeed gives $Q = |\alpha|$ which is a half odd integer. However, in this case, the gauge transformation (5.19) does not exist, so that one cannot interpret this $Q$ as the winding number.

## 6. $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ vs $2+1$ SYM on $R \times S^2$

### 6.1 Embedding of $\mathrm{SYM}_{R\times S^3/Z_k}$ into $\mathrm{SYM}_{R\times S^2}$

In this subsection, we prove the prediction 2) for the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$. According to the prediction 2), the theory around the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ with $U(N)$ gauge group is equivalent to the theory around the vacuum (3.11) of $\mathrm{SYM}_{R\times S^2}$ with the relation (3.10) if a single period is extracted after the periodicity is imposed.

In (5.5), by setting $\alpha_s = sk$, $N_s = N$ and making $s$ run from $-\infty$ to $\infty$, we obtain the theory around the vacuum (3.11) of $\mathrm{SYM}_{R\times S^2}$. Then, the monopole charge $q_{st}$ takes the form

$$q_{st} = \frac{k}{2}(s-t), \tag{6.1}$$

which depends only on $s-t$. This fact enables us to impose the following condition on the blocks of the fields in (5.5):

$$\begin{aligned} X^{(s+1,t+1)} &= X^{(s,t)}, \qquad A_0^{(s+1,t+1)} = A_0^{(s,t)}, \\ \vec{Y}^{(s+1,t+1)} &= \vec{Y}^{(s,t)}, \qquad \psi^{A(s+1,t+1)} = \psi^{A(s,t)}. \end{aligned} \tag{6.2}$$

Namely, the $(s,t)$ blocks of the fields depends only on $s-t$. It is natural to consider that this condition corresponds to the periodicity on the gravity side. We show below that this is indeed the case.

The condition for the modes of these fields follows from (6.2):

$$\begin{aligned} x_{ABJm}^{(s+1,t+1)} &= x_{ABJm}^{(s,t)}, \qquad b_{Jm}^{(s+1,t+1)} = b_{Jm}^{(s,t)}, \\ y_{Jm\rho}^{(s+1,t+1)} &= y_{Jm\rho}^{(s,t)}, \qquad \psi_{Jm\kappa}^{A(s,t)} = \psi_{Jmq\kappa}^{A(s,t)}. \end{aligned} \tag{6.3}$$

This condition allows us to rewrite the modes as

$$\begin{aligned} x_{ABJm}^{(s,t)} &= x_{ABJmq_{st}}, \qquad b_{Jm}^{(s,t)} = b_{Jmq_{st}}, \\ y_{Jm\rho}^{(s,t)} &= y_{Jmq_{st}\rho}, \qquad \psi_{Jm\kappa}^{A(s,t)} = \psi_{Jmq_{st}\kappa}^{A}, \end{aligned} \tag{6.4}$$

Note that every mode is an $N \times N$ matrix.

By using (6.1) and (6.4), we rewrite (5.8). Here we show calculation of some terms in (5.8) as examples. We first consider in $S_{R\times S^2}^{free}$

$$\sum_{s,t} \sum_{J\geq|q_{st}|} \sum_{m=-J}^{J} \left(J+\frac{1}{2}\right)^2 x_{ABJm}^{(s,t)\dagger} x_{ABJm}^{(s,t)}. \tag{6.5}$$

We set $s-t=n,\ \ s=l$ so that $n,\ l$ take integers. We can rewrite (6.5) as

$$\sum_{l} \sum_{n} \sum_{J\geq|\frac{k}{2}n|} \sum_{m=-J}^{J} \left(J+\frac{1}{2}\right)^2 x_{Jm\frac{k}{2}n}^{AB\dagger} x_{Jm\frac{k}{2}n}^{AB}. \tag{6.6}$$

Moreover, by setting $\frac{k}{2}n=\tilde{m}$, we obtain

$$\sum_{l} \sum_{J=0}^{\infty} \sum_{m=-J}^{J} \sum_{\tilde{m}=-J}^{J} \Big|_{\tilde{m}\in\frac{k}{2}\boldsymbol{Z}} \left(J+\frac{1}{2}\right)^2 x_{Jm\tilde{m}}^{AB\dagger} x_{Jm\tilde{m}}^{AB}. \tag{6.7}$$

We next consider in $S_{R\times S^2}^{int}$

$$\sum_{s,t,u} \sum_{J_1\geq|q_{st}|,m_1} \sum_{J_2\geq|q_{tu}|,m_2} \sum_{J_3\geq|q_{us}|,m_3}$$

$$\mathcal{C}_{J_1m_1q_{st}\ J_2m_2q_{tu}\ J_3m_3q_{us}} \partial_0 x_{ABJ_1m_1q_{st}} (b_{J_2m_2q_{tu}} x_{J_3m_3q_{us}}^{AB} - x_{J_2m_2q_{tu}}^{AB} b_{J_3m_3q_{us}}). \tag{6.8}$$

In (6.8), we set $s-t=n,\ t-u=p,\ t=l$ in the first term and $s-t=n,\ u-s=p,\ s=l$ in the second term, so that $n,p,l$ take integers. We also make exchanges for dummy variables in the second term as $J_2 \leftrightarrow J_3,\ m_2 \leftrightarrow m_3$. Then we can rewrite (6.8) as

$$\sum_{l,n,p} \sum_{J_1\geq|\frac{k}{2}n|,m_1} \sum_{J_2\geq|\frac{k}{2}p|,m_2} \tag{6.9}$$

$$\sum_{J_3\geq|\frac{k}{2}(n+p)|,m_3} \mathcal{C}_{J_1 m_1 \frac{k}{2}n\ J_2 m_2 \frac{k}{2}p\ J_3 m_3 \frac{k}{2}(-n-p)} \partial_0 x_{ABJ_1 m_1 \frac{k}{2}n} [b_{J_2 m_2 \frac{k}{2}p}, x^{AB}_{J_3 m_3 \frac{k}{2}(-n-p)}]. \tag{6.10}$$

We further set $\frac{k}{2}n = \tilde{m}_1,\ \frac{k}{2}p = \tilde{m}_2,\ \frac{k}{2}(-n-p) = \tilde{m}_3$, and obtain

$$\sum_l \sum_{J_1=0}^{\infty} \sum_{m_1,\tilde{m}_1=-J_1}^{J_1} \sum_{J_2=0}^{\infty} \sum_{m_2,\tilde{m}_2=-J_2}^{J_2} \sum_{J_3=0}^{\infty} \sum_{m_3,\tilde{m}_3=-J_3}^{J_3} \Bigg|_{\tilde{m}_1,\tilde{m}_2,\tilde{m}_3\in\frac{k}{2}\boldsymbol{Z}}$$
$$\mathcal{C}_{J_1 m_1 \tilde{m}_1\ J_2 m_2 \tilde{m}_2\ J_3 m_3 \tilde{m}_3} \partial_0 x_{ABJ_1 m_1 \tilde{m}_1} [b_{J_2 m_2 \tilde{m}_2}, x^{AB}_{J_3 m_3 \tilde{m}_3}]. \tag{6.11}$$

We can easily rewrite the other terms in (5.8) in the same way. There appears in common the overall factor $\sum_l$ in all the terms of the rewritten form of (5.8).

In appendix G, we give the mode expansion of the theory around the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ (G.1), which we obtained in our previous publication [31]. In the rewritten form of (5.8) obtained above, we make the following identifications

$$\begin{aligned} b_{Jm\tilde{m}} &= B_{Jm\tilde{m}}, \quad y_{Jm\tilde{m}\rho} = A_{Jm\tilde{m}\rho}, \\ x^{AB}_{Jm\tilde{m}} &= X^{AB}_{Jm\tilde{m}}, \quad \psi^A_{Jm\tilde{m}\kappa} = \Psi^A_{Jm\tilde{m}\kappa} \end{aligned} \tag{6.12}$$

and input the relation (3.10). Moreover, we divide this rewritten form by the overall factor $\sum_l$. This procedure corresponds to extracting a single period. Then, it is easy to see that this rewritten form of (5.8) coincides with (G.1).[3] Thus we have completed the proof of the prediction 2) for the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$.

The configuration (3.11), the condition (6.2) and the procedure of dividing by $\sum_l$ physically mean that a circle with the radius$\sim k$ is constructed in the $\Phi$ direction and the $(s,t)$ block of the fields is interpreted as the winding mode around the circle with the winding number $s-t$. We have reinterpreted the winding number $s-t$ as the Kaluza-Klein momentum $\frac{k}{2}(s-t)$ on a circle with the radius$\sim \frac{1}{k}$. This is similar to Taylor's prescription for the compactification (the T-duality) in matrix models [8]. The difference between our prescription and Taylor's is the existence of the nontrivial gauge fields in (3.11), which makes a nontrivial fibration of the circle over $S^2$ rather than a direct product $S^2\times S^1$ so that $S^3/Z_k$ is realized.

### 6.2 $S^3$ from three matrices

Combining the result in section 5.1 with that in section 6.1 leads us to conclude that the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ with gauge group $U(N)$ is embedded in PWMM. The

[3] More precisely, the terms proportional to $\mu$ differ in signature. However, this difference can be compensated by the parity transformation, so that it does not matter.



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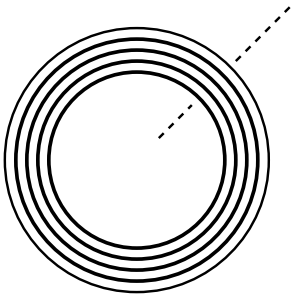

**Figure 6:** $S^3/Z_k$ is realized through a stack of fuzzy spheres. Each circle represents $N$ coincident fuzzy spheres.

corresponding vacuum configuration of PWMM is $Y_i = -\mu L_i$, where

$$L_i = \begin{pmatrix} \ddots & & & & & & \\ & L_i^{[j_{s-1}]} & N & & & & \\ & & \ddots & L_i^{[j_{s-1}]} & & & \\ & & & & L_i^{[j_s]} & N & \\ & & & & & \ddots & L_i^{[j_s]} \\ & & & & & & & L_i^{[j_{s+1}]} & N \\ & & & & & & & & \ddots & L_i^{[j_{s+1}]} \\ & & & & & & & & & \ddots \end{pmatrix} \tag{6.13}$$

with $2j_s + 1 = N_0 + ks$. $s$ runs from $-\infty$ to $\infty$ and the following periodicity for the fluctuations of the fields around the vacuum (6.13) is imposed:

$$\vec{Y}^{(s+1,t+1)} = \vec{Y}^{(s,t)}, \quad X_m^{(s+1,t+1)} = X_m^{(s,t)}, \quad \lambda^{(s+1,t+1)} = \lambda^{(s,t)}. \tag{6.14}$$

The vacuum (6.13) is interpreted as a stack of infinitely many sets of $N$ coincident fuzzy spheres (See figure 6 ). Note that the $N_0 \to \infty$ limit must be taken from the beginning in order for the configuration (6.13) to be realized.

It is interesting that $S^3/Z_k$ is realized by the three matrices, $Y_1$, $Y_2$, $Y_3$. It is well-known that fuzzy sphere is realized by three matrices through the $SU(2)$ algebra and in the continuum limit an ordinary $S^2$ is realized with one of three directions remained on $S^2$ as a Higgs field. In the present case, the Higgs field is utilized to make the $U(1)$ bundle on $S^2$. In particular, in the $k = 1$ case, one realizes $S^3$ by the three matrices and obtains from PWMM $\mathcal{N} = 4$ SYM on $R \times S^3$, which is important in the AdS/CFT context, namely, dual

to $AdS_5 \times S^5$ in the global coordinates. In this case, the $SU(2|4)$ symmetry is enhanced to the $SU(2,2|4)$ symmetry.

## 7. Summary and outlook

In this paper, we show that every vacuum of $\mathrm{SYM}_{R\times S^2}$ is embedded in PWMM and the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ is embedded in $\mathrm{SYM}_{R\times S^2}$. This is predicted from the gravity duals through Lin-Maldacena's method. Our results serve as a nontrivial check of the gauge/gravity correspondence for the theories with $SU(2|4)$ symmetry. As by-products, we reveal the relationships among the spherical harmonics on $S^3$, the monopole harmonics and the fuzzy sphere harmonics, and extend an extension of the compactification (T-duality) in matrix models a la Taylor to that on spheres.

We treated only embedding of the trivial vacuum of $\mathrm{SYM}_{R\times S^3/Z_k}$ into $\mathrm{SYM}_{R\times S^2}$. Indeed, we have the vacuum configurations in $\mathrm{SYM}_{R\times S^2}$ that would give the theories around the nontrivial vacua of $\mathrm{SYM}_{R\times S^3/Z_k}$. It is important to prove the prediction 2) for the nontrivial vacua.

It is interesting to extend the T-duality in matrix models in this paper, which realizes $S^3/Z_k$ as an $S^1$ fibration over $S^2$, to other fiber bundles and to obtain a general recipe for such T-duality in matrix models.

$\mathrm{SYM}_{R\times S^3/Z_k}$ with $k = 1$ is nothing but $\mathcal{N} = 4$ SYM on $R \times S^3$, which has the unique trivial vacuum and whose symmetry group is enhanced to $SU(2,2|4)$. The gravity dual of this theory is $AdS_5 \times S^5$. Hence as mentioned in section 6.2, our results tell that $\mathcal{N} = 4$ SYM on $R \times S^3$ which is a gauge theory in a typical example of the AdS/CFT correspondence is embedded in PWMM. However, this does not mean that we have obtained a matrix model that regularizes $\mathcal{N} = 4$ SYM on $R \times S^3$ preserving gauge symmetry and supersymmetry and in principle enables us to perform a numerical simulation for the AdS/CFT correspondence. Indeed, in the T-duality, we need to consider matrices with infinite size. Presumably, by referring to the work [43], we can make the size of matrices finite with a part of supersymmetry preserved and obtain a lattice gauge theory with few parameters to be fine-tuned for $\mathcal{N} = 4$ SYM on $R \times S^3$.

We hope to report progress in the above projects in the near future.

**Note added.** While we are writing the manuscript, we are informed that Aoki et al. are preparing for a publication [44], which has some overlap with section 4.3 of the present paper.

## Acknowledgments

We would like to thank H. Aoki for discussions and informing us of the results in [44] before publication. We are also grateful to H. Kawai for discussions. We thank the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the YITP workshop YITP-W-06-11 on "String Theory and Quantum Field Theory" were useful to complete this work. The work of Y.T. is supported in part by The 21st Century COE Program "Towards a New Basic Science; Depth and Synthesis." The work of A.T. is supported in

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## A. Some conventions

In this appendix, we describe some conventions which we follow in the present paper.

We use the following metric for $R \times S^3$:

$$ds^2_{R\times S^3} = -dt^2 + \frac{1}{\mu^2}(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2), \tag{A.1}$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, $0 \leq \psi < 4\pi$, and the radius of $S^3$ is $\frac{2}{\mu}$. The nonvanishing components of the vierbeins and the spin connections are

$$\begin{aligned} &e^1_\theta = \mu^{-1}, \quad e^2_\phi = \mu^{-1}, \quad e^3_\phi = \mu^{-1}\cos\theta, \quad e^3_\psi = \mu^{-1}, \\ &\omega_{12} = -\omega_{21} = -\frac{1}{2}\cos\theta d\phi + \frac{1}{2}d\psi, \quad \omega_{23} = -\omega_{32} = -\frac{1}{2}d\theta, \quad \omega_{31} = -\omega_{13} = -\frac{1}{2}\sin\theta d\phi. \end{aligned} \tag{A.2}$$

We use the following metric for $R \times S^2$:

$$ds^2_{R\times S^2} = -dt^2 + \frac{1}{\mu^2}(d\theta^2 + \sin^2\theta d\phi^2). \tag{A.3}$$

Here the radius of $S^2$ is $\frac{1}{\mu}$. The nonvanishing components of the dreibeins and the spin connections are

$$b^1_\theta = \mu^{-1}, \quad b^2_\phi = \mu^{-1}\sin\theta, \quad k_{12} = -k_{21} = -\cos\theta d\phi. \tag{A.4}$$

It is convenient for the mode expansions to rewrite the actions in the $SU(4)$ symmetric form. The 10-dimensional Lorentz group has been decomposed as $SO(9,1) \supset SO(3,1) \times SO(6)$. We identify $SO(6)$ with $SU(4)$. We use $A, B = 1, 2, 3, 4$ as the indices of $\mathbf{4}$ in $SU(4)$ while we have used $m, n = 4, \cdots, 9$ as the indices of $\mathbf{6}$ in $SO(6)$. The $SO(6)$ vector, $\mathbf{6}$, corresponds to the antisymmetric tensor of $\mathbf{4}$ in $SU(4)$. The $SO(6)$ and $SU(4)$ basis are related as

$$\begin{aligned} &X_{i4} = \frac{1}{2}(X_{i+3} + iX_{i+6}) \quad (i = 1, 2, 3), \\ &X_{AB} = -X_{BA}, \quad X^{AB} = -X^{BA} = X^\dagger_{AB}, \quad X^{AB} = \frac{1}{2}\epsilon^{ABCD}X_{CD}. \end{aligned} \tag{A.5}$$

Similar identities hold for the gamma matrices:

$$\Gamma^{i4} = \frac{1}{2}(\Gamma^{i+3} - i\Gamma^{i+6}), \quad \text{etc.} \tag{A.6}$$

The 10-dimensional gamma matrices are decomposed as

$$\Gamma^a = \gamma^a \otimes 1_8, \quad \Gamma^{AB} = \gamma_5 \otimes \begin{pmatrix} 0 & -\tilde{\rho}^{AB} \\ \rho^{AB} & 0 \end{pmatrix} = -\Gamma^{BA}, \tag{A.7}$$

where $\gamma^a$ is the 4-dimensional gamma matrix, satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and $\gamma_5 = i\gamma^0\gamma^1 \times \gamma^2\gamma^3$. $\Gamma^{AB}$ satisfies $\{\Gamma^{AB}, \Gamma^{CD}\} = \epsilon^{ABCD}$, and $\rho^{AB}$ and $\tilde{\rho}^{AB}$ are defined by

$$(\rho^{AB})_{CD} = \delta^A_C\delta^B_D - \delta^A_D\delta^B_C, \quad (\tilde{\rho}^{AB})^{CD} = \epsilon^{ABCD}. \tag{A.8}$$

The charge conjugation matrix and the chirality matrix are given by

$$C_{10} = C_4 \otimes \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}, \qquad \Gamma^{11} = \Gamma^0 \cdots \Gamma^9 = \gamma_5 \otimes \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}, \tag{A.9}$$

where $(\Gamma^{a,m})^T = -C_{10}^{-1}\Gamma^{a,m}C_{10}$ and $C_4$ is the charge conjugation matrix in 4 dimensions. The Majorana-Weyl spinor in 10 dimensions is decomposed as

$$\lambda = \Gamma_{11}\lambda = \begin{pmatrix} \lambda^A_+ \\ \lambda_{-A} \end{pmatrix}, \tag{A.10}$$

where $\lambda_{-A}$ is the charge conjugation of $\lambda^A_+$:

$$\lambda_{-A} = (\lambda^A_+)^c = C_4(\bar{\lambda}_{+A})^T, \qquad \gamma_5\lambda_\pm = \pm\lambda_\pm. \tag{A.11}$$

We further fix the forms of 4-dimensional gamma matrices:

$$\gamma^a = \begin{pmatrix} 0 & i\sigma^a \\ i\bar{\sigma}^a & 0 \end{pmatrix}, \tag{A.12}$$

where $\sigma^0 = -1_2$ and $\sigma^i \quad (i = 1, 2, 3)$ are the Pauli matrices. $\bar{\sigma}^0 = \sigma^0$ and $\bar{\sigma}^i = -\sigma^i$. In this convention,

$$\gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \qquad C_4 = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}. \tag{A.13}$$

We introduce a two-component spinor:

$$\lambda^A_+ = \begin{pmatrix} \psi^A \\ 0 \end{pmatrix}. \tag{A.14}$$

Using the $SU(4)$ symmetric notation, one can rewrite the actions (2.1), (2.21) and (2.22) as follows:

$$\begin{aligned} S_{R\times S^3} = \frac{1}{g^2_{R\times S^3}} \int dt \frac{d\Omega_3}{(\mu/2)^3} \,\mathrm{Tr}\Big( & -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}D_aX_{AB}D^aX^{AB} - \frac{1}{2}X_{AB}X^{AB} \\ & + i\psi^\dagger_A D_0\psi^A + i\psi^\dagger_A\sigma^i D_i\psi^A + \psi^\dagger_A\sigma^2[X^{AB}, (\psi^\dagger_B)^T] \\ & - \psi^{AT}\sigma^2[X_{AB}, \psi^B] + \frac{1}{4}[X_{AB}, X_{CD}][X^{AB}, X^{CD}]\Big), \end{aligned} \tag{A.15}$$

$$S_{R\times S^2} = \frac{1}{g^2_{R\times S^2}} \int dt \frac{d\Omega_2}{\mu^2} \mathrm{Tr}\Big( \frac{1}{2}(D_0\vec{Y} - i\mu \vec{L}^{(0)} A_0)^2 - \frac{1}{2}\vec{\mathcal{Z}}^2 + \frac{1}{2} D_0 X_{AB} D_0 X^{AB}$$
$$+ \frac{1}{2}\vec{\mathcal{L}} X_{AB} \cdot \vec{\mathcal{L}} X^{AB} - \frac{\mu^2}{8} X_{AB} X^{AB} + \frac{1}{4}[X_{AB}, X_{CD}][X^{AB}, X^{CD}]$$
$$+ i\psi^\dagger_A D_0 \psi^A - \psi^\dagger_A \vec{\sigma} \cdot \vec{\mathcal{L}} \psi^A - \frac{3\mu}{4}\psi^\dagger_A \psi^A + \psi^\dagger_A \sigma^2 [X^{AB}, (\psi^\dagger_B)^T]$$
$$- \psi^{AT} \sigma^2 [X_{AB}, \psi^B] \Big), \tag{A.16}$$

$$S_{PW} = \frac{1}{g^2_{PW}} \int \frac{dt}{\mu^2} \mathrm{Tr}\Big( \frac{1}{2}(D_0 Y_i)^2 - \frac{1}{2}(\mu Y_i - \frac{i}{2}\epsilon_{ijk}[Y_j, Y_k])^2 + \frac{1}{2} D_0 X_{AB} D_0 X^{AB}$$
$$- \frac{\mu^2}{8} X_{AB} X^{AB} + \frac{1}{2}[Y_i, X_{AB}][Y_i, X^{AB}] + \frac{1}{4}[X_{AB}, X_{CD}][X^{AB}, X^{CD}]$$
$$+ i\psi^\dagger_A D_0 \psi^A - \frac{3\mu}{4}\psi^\dagger_A \psi^A + \psi^\dagger_A \sigma^i [Y_i, \psi^A] + \psi^\dagger_A \sigma^2 [X^{AB}, (\psi^\dagger_B)^T]$$
$$- \psi^{AT} \sigma^2 [X_{AB}, \psi^B] \Big). \tag{A.17}$$

## B. The plane wave matrix model

In this appendix, we give the relationship between the action (2.22) and the conventional form of the action of the plane wave matrix model in the literature. We introduce another representation of the 10-dimensional gamma matrices as follows:

$$\Gamma^0 = 1_{16} \otimes (-i)\sigma^2, \quad \Gamma^{\hat{M}} = \gamma^{\hat{M}} \otimes \sigma^3, \tag{B.1}$$

where $\gamma^{\hat{M}}$ is the $SO(9)$ gamma matrix, which is a $16 \times 16$ real symmetric matrix, and $\hat{M} = (i, m)$. In this representation, the charge conjugation matrix is $C_{10} = \Gamma^0$, and $\Gamma^{11} = 1_{16} \otimes \sigma^1$. Then the Majorana-Weyl spinor $\lambda$ is represented as

$$\lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi \\ \Psi \end{pmatrix}, \tag{B.2}$$

where $\Psi$ is a real 16-components spinor. We make a redefinition, $Y^i \to X^i$. We also rescale the fields, the coupling constant and the time as follows:

$$A_0 \to -3\mu g A_0, \quad X^{\hat{M}} \to -\mu g X^{\hat{M}}, \quad \Psi \to -\sqrt{3}\mu^{\frac{3}{2}} g \Psi,$$
$$g \to \sqrt{3\mu} g, \quad t \to 3\mu t. \tag{B.3}$$

We finally obtain from (2.22)

$$S_{PWMM} = \int dt\, \mathrm{Tr}\left( \frac{1}{2} D_0 X^{\hat{M}} D_0 X^{\hat{M}} - \frac{1}{18} X^i X^i - \frac{1}{72} X^m X^m - \frac{ig}{18} \epsilon_{ijk} X^i [X^j, X^k] \right.$$
$$\left. + \frac{g^2}{36} [X^{\hat{M}}, X^{\hat{N}}]^2 + \frac{i}{2}\Psi^\dagger D_0 \Psi - \frac{i}{8}\Psi^\dagger \gamma^{123} \Psi + \frac{g}{6}\Psi^\dagger \gamma^{\hat{M}} [X^{\hat{M}}, \Psi] \right), \tag{B.4}$$

where $D_0 = \partial_t + ig[A_0, \,]$. This is the conventional form of the action of the plane wave matrix model seen in the literature.

## C. Supersymmetry transformations

In this appendix, we give the supersymmetry transformation rules for the theories with $SU(2|4)$ symmetry.

First, the action of PWMM (2.22) is invariant under the following supersymmetry transformations:

$$\begin{aligned}
&\delta A^0 = -i\bar{\eta}\Gamma^0\lambda, \\
&\delta\vec{Y} = -i\bar{\eta}\vec{\Gamma}\lambda, \\
&\delta X^m = -i\bar{\eta}\Gamma^m\lambda, \\
&\delta\lambda = D_0 Y^i\Gamma^{0i}\eta + D_0 X^m\Gamma^{0m}\eta + \mu Y^i\Gamma^{i123}\eta - \frac{\mu}{2}X^m\Gamma^{m123}\eta \\
&\qquad -\frac{i}{2}[Y^i, Y^j]\Gamma^{ij}\eta - i[Y^i, X^m]\Gamma^{im}\eta - \frac{i}{2}[X^m, X^n]\Gamma^{mn}\eta,
\end{aligned} \tag{C.1}$$

where the parameter $\eta$ is a 10-dimensional Majorana-Weyl spinor which satisfies $\partial_0\eta = -\frac{\mu}{4}\Gamma^{0123}\eta$. Then, the theory has 16 supercharges.

Next, the action of $\mathrm{SYM}_{R\times S^2}$ (2.21) is invariant under the following transformations:

$$\begin{aligned}
&\delta A^0 = -i\bar{\eta}\Gamma^0\lambda, \\
&\delta\vec{Y} = -i\bar{\eta}\vec{\Gamma}\lambda, \\
&\delta X^m = -i\bar{\eta}\Gamma^m\lambda, \\
&\delta\lambda = D_0 Y^i\Gamma^{0i}\eta + D_0 X^m\Gamma^{0m}\eta - \frac{\mu}{2}X^m\Gamma^{m123}\eta + i\mathcal{L}_i X^m\Gamma^{im} \\
&\qquad -\frac{i}{2}[X^m, X^n]\Gamma^{mn}\eta + \frac{1}{2}\epsilon_{ijk}\mathcal{Z}_i\Gamma^{jk}\eta - i\mu L_i^{(0)}A_0\Gamma^{0i}\eta.
\end{aligned} \tag{C.2}$$

Again, $\eta$ is a 10-dimensional Majorana-Weyl spinor which satisfies $\partial_0\eta = -\frac{\mu}{4}\Gamma^{0123}\eta$. The theory also has 16 supercharges.

Finally, the transformation rule for the original $\mathcal{N} = 4$ SYM on $R\times S^3$ (2.1) is as follows:

$$\begin{aligned}
&\delta A_a = i\bar{\lambda}\Gamma_a\epsilon, \\
&\delta X_m = i\bar{\lambda}\Gamma_m\epsilon, \\
&\delta\lambda = \left[\frac{1}{2}F_{ab}\Gamma^{ab} + D_a X_m\Gamma^{am} - \frac{1}{2}X_m\Gamma^{ma}\nabla_a - \frac{i}{2}[X_m, X_n]\Gamma^{mn}\right]\epsilon.
\end{aligned} \tag{C.3}$$

In this case, the parameter $\epsilon$ is a conformal Killing spinor on $R\times S^3$. In order to write down the conformal Killing spinor equation, we decompose $\epsilon$ into the 4-dimensional Majorana-Weyl spinors as

$$\epsilon = \begin{pmatrix} \epsilon_+^A \\ \epsilon_{-A} \end{pmatrix}, \tag{C.4}$$

where $\epsilon_+^A$ and $\epsilon_{-A}$ are the 4-dimensional Majorana-Weyl spinors, and $\epsilon_{-A}$ is the charge conjugation of $\epsilon_+^A$ (see Appendix A). Then, the conformal Killing spinor equation on $R\times S^3$ is written as

$$\nabla_a\epsilon_+^A = \pm\frac{i}{2}\gamma_a\gamma^0\epsilon_+^A, \quad \gamma_5\epsilon_+^A = \epsilon_+^A. \tag{C.5}$$

A general solution of above equation has four real degrees of freedom for each sign, and there are four SU(4) indices, so that the original 10-dimensional parameter $\epsilon$ possess 32 real degrees of freedom. In $\mathrm{SYM}_{R\times S^3/Z_k}$, there remain only supersymmetries caused by the conformal Killing spinors that satisfy the lower sign of (C.5), so that only 16 supercharges survive.

## D. Useful formulae for representations of $SU(2)$

In this appendix, we gather some useful formulae concerning the representations of $SU(2)$, most of which are found in [42]. The relationship between the Clebsch-Gordan coefficient and the $3-j$ symbol is

$$\begin{pmatrix} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{J_3+m_3+2J_1} \frac{1}{\sqrt{2J_3+1}}\, C^{J_3 m_3}_{J_1\, -m_1\; J_2\, -m_2}. \tag{D.1}$$

The Clebsch-Gordan coefficient possesses the following symmetries:

$$\begin{aligned} C^{J_3 m_3}_{J_1 m_1\; J_2 m_2} &= (-1)^{m_1+m_2-m_3} C^{J_3 m_3}_{J_2 m_2\; J_1 m_1} \\ &= (-1)^{J_1-m_1}\sqrt{\frac{2m_3+1}{2m_2+1}} C^{J_2\, -m_2}_{J_1 m_1\; J_3\, -m_3} = (-1)^{J_1-m_1}\sqrt{\frac{2m_3+1}{2m_2+1}} C^{J_2\, m_2}_{J_3 m_3\; J_1\, -m_1} \\ &= (-1)^{J_2+m_2}\sqrt{\frac{2m_3+1}{2m_1+1}} C^{J_1\, -m_1}_{J_3\, -m_3\; J_2 m_2} = (-1)^{J_2+m_2}\sqrt{\frac{2m_3+1}{2m_1+1}} C^{J_1\, m_1}_{J_2\, -m_2\; J_3 m_3}, \\ C^{J_3 m_3}_{J_1 m_1\; J_2 m_2} &= (-1)^{m_1+m_2\, -m_3} C^{J_3\, -m_3}_{J_1\, -m_1\; J_2\, -m_2}. \end{aligned} \tag{D.2}$$

The recursion relation for the Clebsch-Gordan coefficient is

$$\begin{aligned} &\sqrt{(c\pm\gamma)(c\mp\gamma+1)}\, C^{c\gamma\mp 1}_{a\alpha\; b\beta} = \\ &\quad = \sqrt{(a\mp\alpha)(a\pm\alpha+1)}\, C^{c\gamma}_{a\alpha\pm 1\; b\beta} + \sqrt{(b\mp\beta)(b\pm\beta+1)}\, C^{c\gamma}_{a\alpha\; b\beta\pm 1}. \end{aligned} \tag{D.3}$$

In sections 4, we frequently use a summation formula for the Clebsch-Gordan coefficient,

$$\sum_{m_1 m_2} C^{J_3 m_3}_{J_1 m_1\; J_2 m_2} C^{J_3' m_3'}_{J_1 m_1\; J_2 m_2} = \delta_{J_3 J_3'}\delta_{m_3 m_3'}, \tag{D.4}$$

$$\sum_{\alpha\beta\delta} C^{c\gamma}_{a\alpha\; b\beta} C^{e\epsilon}_{d\delta\; b\beta} C^{d\delta}_{a\alpha\; f\varphi} = (-1)^{b+c+d+f}\sqrt{(2c+1)(2d+1)}\, C^{e\epsilon}_{c\gamma\; f\varphi} \begin{Bmatrix} a & b & c \\ e & f & d \end{Bmatrix}, \tag{D.5}$$

$$\begin{aligned} &\sum_{\beta\gamma\epsilon\varphi} C^{a\alpha}_{b\beta\; c\gamma} C^{d\delta}_{e\epsilon\; f\varphi} C^{b\beta}_{e\epsilon\; g\eta} C^{c\gamma}_{f\varphi\; j\mu} \\ &= \sum_{k\kappa} \sqrt{(2b+1)(2c+1)(2d+1)(2k+1)}\, C^{k\kappa}_{g\eta\; j\mu} C^{a\alpha}_{d\delta\; k\kappa} \begin{Bmatrix} a & b & c \\ d & e & f \\ k & g & j \end{Bmatrix}. \end{aligned} \tag{D.6}$$

In section 4, the following identity is often used:

$$\langle Jm|e^{i\theta J_1}|Jn\rangle^* = (-1)^{-m+n}\langle J\, -m|e^{i\theta J_1}|J\, -n\rangle. \tag{D.7}$$

In section 5, we use a formula for the asymptotic relations between the $6-j$ symbols and the $3-j$ symbols. If $R \gg 1$, one obtains

$$\begin{Bmatrix} a & b & c \\ d+R & e+R & f+R \end{Bmatrix} \approx \frac{(-1)^{a+b+c+2(d+e+f)}}{\sqrt{2R}} \begin{pmatrix} a & b & c \\ e-f & f-d & d-e \end{pmatrix}. \tag{D.8}$$

## E. Vertex coefficients

In this appendix, we give expressions for the vertex coefficients we defined in section 4. These expressions are obtained by using the formula (4.15). In the following, $Q \equiv J + \frac{(1+\rho)\rho}{2}$, $\tilde{Q} \equiv J - \frac{(1-\rho)\rho}{2}$, $U \equiv J + \frac{1+\kappa}{4}$ and $\tilde{U} \equiv J + \frac{1-\kappa}{4}$. Suffices on these variables must be understood appropriately.

$$\mathcal{C}^{J_1 m_1 \tilde{m}_1}_{J_2 m_2 \tilde{m}_2\ J_3 m_3 \tilde{m}_3} = \sqrt{\frac{(2J_2+1)(2J_3+1)}{2J_1+1}} C^{J_1 m_1}_{J_2 m_2\ J_3 m_3} C^{J_1 \tilde{m}_1}_{J_2 \tilde{m}_2\ J_3 \tilde{m}_3}, \tag{E.1}$$

$$\begin{aligned} \mathcal{D}^{J m \tilde{m}}_{J_1 m_1 \tilde{m}_1 \rho_1\ J_2 m_2 \tilde{m}_2 \rho_2} &= (-1)^{\frac{\rho_1+\rho_2}{2}+1} \sqrt{3(2J_1+1)(2J_1+2\rho_1^2+1)(2J_2+1)(2J_2+2\rho_2^2+1)} \\ &\quad \times \begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ J & J & 0 \end{Bmatrix} C^{Jm}_{Q_1 m_1\ Q_2 m_2} C^{J\tilde{m}}_{\tilde{Q}_1 \tilde{m}_1\ \tilde{Q}_2 \tilde{m}_2}, \end{aligned} \tag{E.2}$$

$$\begin{aligned} &\mathcal{E}_{J_1 m_1 \tilde{m}_1 \rho_1\ J_2 m_2 \tilde{m}_2 \rho_2\ J_3 m_3 \tilde{m}_3 \rho_3} \\ &= \sqrt{6(2J_1+1)(2J_1+2\rho_1^2+1)(2J_2+1)(2J_2+2\rho_2^2+1)(2J_3+1)(2J_3+2\rho_3^2+1)} \\ &\quad \times (-1)^{-\frac{\rho_1+\rho_2+\rho_3+1}{2}} \begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ Q_3 & \tilde{Q}_3 & 1 \end{Bmatrix} \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \tilde{Q}_1 & \tilde{Q}_2 & \tilde{Q}_3 \\ \tilde{m}_1 & \tilde{m}_2 & \tilde{m}_3 \end{pmatrix}, \end{aligned} \tag{E.3}$$

$$\mathcal{F}^{J_1 m_1 \tilde{m}_1 \kappa_1}_{J_2 m_2 \tilde{m}_2 \kappa_2\ J m \tilde{m}} = \sqrt{2(2J+1)^2(2J_2+1)(2J_2+2)} \begin{Bmatrix} U_1 & \tilde{U}_1 & \frac{1}{2} \\ U_2 & \tilde{U}_2 & \frac{1}{2} \\ J & J & 0 \end{Bmatrix} C^{U_1 m_1}_{U_2 m_2\ J m} C^{\tilde{U}_1 \tilde{m}_1}_{\tilde{U}_2 \tilde{m}_2\ J \tilde{m}}, \tag{E.4}$$

$$\begin{aligned} \mathcal{G}^{J_1 m_1 \tilde{m}_1 \kappa_1}_{J_2 m_2 \tilde{m}_2 \kappa_2\ J m \tilde{m} \rho} &= (-1)^{\frac{\rho}{2}} \sqrt{6(2J_2+1)(2J_2+2)(2J+1)(2J+2\rho^2+1)} \\ &\quad \times \begin{Bmatrix} U_1 & \tilde{U}_1 & \frac{1}{2} \\ U_2 & \tilde{U}_2 & \frac{1}{2} \\ Q & \tilde{Q} & 1 \end{Bmatrix} C^{U_1 m_1}_{U_2 m_2\ Q m} C^{\tilde{U}_1 \tilde{m}_1}_{\tilde{U}_2 \tilde{m}_2\ \tilde{Q} \tilde{m}}. \end{aligned} \tag{E.5}$$

## F. Vertex coefficients of the fuzzy sphere harmonics

In this appendix, we give expressions for the traces of various three fuzzy sphere harmonics which are defined in section 4.3.

$$\begin{aligned} &\hat{\mathcal{C}}^{J_1 m_1 (j' j)}_{J_2 m_2 (j' j'')\ J_3 m_3 (j'' j)} \\ &= (-1)^{J_1+2J_3-\zeta-\zeta'-\zeta''+j-j'} \sqrt{N_0(2J_2+1)(2J_3+1)} C^{J_1 m_1}_{J_2 m_2\ J_3 m_3} \begin{Bmatrix} J_1 & J_2 & J_3 \\ j'' & j & j' \end{Bmatrix}, \end{aligned} \tag{F.1}$$

$$\hat{\mathcal{D}}^{Jm(j'j)}_{J_1m_1(j'j'')\rho_1\ J_2m_2(j''j)\rho_2}$$
$$= \sqrt{3N_0(2J+1)(2J_1+1)(2J_1+2\rho_1^2+1)(2J_2+1)(2J_2+2\rho_2^2+1)}$$
$$\times(-1)^{\frac{\rho_1+\rho_2}{2}+1+J+2\tilde{Q}_2-\zeta-\zeta'-\zeta''+j-j'}\begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ J & J & 0 \end{Bmatrix} C^{Jm}_{Q_1m_1\ Q_2m_2}\begin{Bmatrix} J & \tilde{Q}_1 & \tilde{Q}_2 \\ j'' & j & j' \end{Bmatrix}, \quad \text{(F.2)}$$

$$\hat{\mathcal{E}}_{J_1m_1(jj')\rho_1\ J_2m_2(j'j'')\rho_2\ J_3m_3(j''j)\rho_3}$$
$$= \sqrt{6N_0(2J_1+1)(2J_1+2\rho_1^2+1)(2J_2+1)(2J_2+2\rho_2^2+1)(2J_3+1)(2J_3+2\rho_3^2+1)}$$
$$\times(-1)^{-\frac{\rho_1+\rho_2+\rho_3+1}{2}-\tilde{Q}_1-\tilde{Q}_2-\tilde{Q}_3-\zeta-\zeta'-\zeta''}\begin{Bmatrix} Q_1 & \tilde{Q}_1 & 1 \\ Q_2 & \tilde{Q}_2 & 1 \\ Q_3 & \tilde{Q}_3 & 1 \end{Bmatrix}\begin{pmatrix} Q_1 & Q_2 & Q_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{Bmatrix} \tilde{Q}_1 & \tilde{Q}_2 & \tilde{Q}_3 \\ j'' & j & j' \end{Bmatrix}, \quad \text{(F.3)}$$

$$\hat{\mathcal{F}}^{J_1m_1(j'j)\kappa_1}_{J_2m_2(j'j'')\kappa_2\ Jm(j''j)}$$
$$= \sqrt{2N_0(2\tilde{U}+1)(2J+1)^2(2J_2+1)(2J_2+2)}$$
$$\times(-1)^{\tilde{U}_1+2J-\zeta-\zeta'-\zeta''+j-j'}\begin{Bmatrix} U_1 & \tilde{U}_1 & \frac{1}{2} \\ U_2 & \tilde{U}_2 & \frac{1}{2} \\ J & J & 0 \end{Bmatrix} C^{U_1m_1}_{U_2m_2\ Jm}\begin{Bmatrix} \tilde{U}_1 & \tilde{U}_2 & J \\ j'' & j & j' \end{Bmatrix}, \quad \text{(F.4)}$$

$$\hat{\mathcal{G}}^{J_1m_1(j'j)\kappa_1}_{J_2m_2(j'j'')\kappa_2\ Jm(j''j)\rho}$$
$$= \sqrt{6N_0(2\tilde{U}_1+1)(2J_2+1)(2J_2+2)(2J+1)(2J+2\rho^2+1)}$$
$$\times(-1)^{\frac{\rho}{2}+\tilde{U}_1+2\tilde{U}-\zeta-\zeta'-\zeta''+j-j'}\begin{Bmatrix} U_1 & \tilde{U}_1 & \frac{1}{2} \\ U_2 & \tilde{U}_2 & \frac{1}{2} \\ Q & \tilde{Q} & 1 \end{Bmatrix} C^{U_1m_1}_{U_2m_2\ Qm}\begin{Bmatrix} \tilde{U}_1 & \tilde{U}_2 & \tilde{Q} \\ j'' & j & j' \end{Bmatrix}. \quad \text{(F.5)}$$

As mentioned in section 4.3, In the $N_0 \to \infty$, these reduce to the vertex coefficients in appendix E.

## G. Mode expansion of SYM$_{R\times S^3/Z_k}$

In this appendix, we describe the mode expansion of the theory around the trivial vacuum of SYM$_{R\times S^3/Z_k}$, which was obtained in our previous publication [31]. The result is

$$S_{R\times S^3/Z_k} = S^{free}_{R\times S^3/Z_k} + S^{int}_{R\times S^3/Z_k},$$
$$S^{free}_{R\times S^3/Z_k} = \frac{16\pi^2}{g^2_{R\times S^3/Z_k}k\mu^3}\int dt \mathrm{Tr}\Big\{\sum_{Jm\tilde{m}}\frac{1}{2}(\partial_0 X^{AB\dagger}_{Jm\tilde{m}}\partial_0 X^{AB}_{Jm\tilde{m}} - \mu^2(J+\frac{1}{2})^2 X^{AB\dagger}_{Jm\tilde{m}}X^{AB}_{Jm\tilde{m}})$$
$$+\sum_{\rho=-1}^{1}\sum_{Jm\tilde{m}}\frac{1}{2}(\partial_0 A^\dagger_{Jm\tilde{m}\rho}\partial_0 A_{Jm\tilde{m}\rho} - \mu^2\rho^2(J+1)^2 A^\dagger_{Jm\tilde{m}\rho}A_{Jm\tilde{m}\rho})$$
$$+\sum_{Jm\tilde{m}}\left(\frac{\mu^2}{2}J(J+1)B^\dagger_{Jm\tilde{m}}B_{Jm\tilde{m}} + i\mu\sqrt{J(J+1)}\partial_0 A^\dagger_{Jm\tilde{m}0}B_{Jm\tilde{m}}\right)$$



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$$+\sum_{\kappa=\pm 1}\sum_{Jm\tilde{m}}\left(i\Psi^{\dagger}_{AJm\tilde{m}\kappa}\partial_0\Psi^{A}_{Jm\tilde{m}\kappa}+\kappa\mu(J+\frac{3}{4})\Psi^{\dagger}_{AJm\tilde{m}\kappa}\Psi^{A}_{Jm\tilde{m}\kappa}\right)\Big\},$$

$$\begin{aligned}
S^{int}_{R\times S^3/Z_k} = \frac{16\pi^2}{g^2_{R\times S^3/Z_k}k\mu^3}\int dt \mathrm{Tr}\Big\{ & -i\mathcal{C}_{Jm\tilde{m}\ J_1m_1\tilde{m}_1\ J_2m_2\tilde{m}_2}\partial_0 X^{J_1m_1\tilde{m}_1}_{AB}[B_{Jm\tilde{m}},X^{AB}_{J_2m_2\tilde{m}_2}] \\
& -\frac{1}{2}\mathcal{C}^{Jm\tilde{m}}_{J_1m_1\tilde{m}_1\ J_2m_2\tilde{m}_2}\mathcal{C}_{Jm\tilde{m}\ J_2m_3\tilde{m}_3\ J_4m_4\tilde{m}_4}[B_{J_1m_1\tilde{m}_1},X^{J_2m_2\tilde{m}_2}_{AB}][B_{J_3m_3\tilde{m}_3},X^{AB}_{J_4m_4\tilde{m}_4}]) \\
& +\mu\sqrt{J_1(J_1+1)}\mathcal{D}_{J_2m_2\tilde{m}_2\ J_1m_1\tilde{m}_10\ Jm\tilde{m}\rho}X^{J_1m_1\tilde{m}_1}_{AB}[A_{Jm\tilde{m}\rho},X^{AB}_{J_2m_2\tilde{m}_2}] \\
& +\frac{1}{2}\mathcal{C}^{Jm\tilde{m}}_{J_1m_1\tilde{m}_1\ J_3m_3\tilde{m}_3}\mathcal{D}_{Jm\tilde{m}\ J_2m_2\tilde{m}_2\rho_2\ J_4m_4\tilde{m}_4\rho_4}[X^{J_1m_1\tilde{m}_1}_{AB},A_{J_2m_2\tilde{m}_2\rho_2}] \\
& [X^{AB}_{J_3m_3\tilde{m}_3},A_{J_4m_4\tilde{m}_4\rho_4}] \\
& +\frac{1}{4}\mathcal{C}^{Jm\tilde{m}}_{J_1m_1\tilde{m}_1\ J_2m_2\tilde{m}_2}\mathcal{C}_{Jm\tilde{m}\ J_3m_3\tilde{m}_3\ J_4m_4\tilde{m}_4}[X^{J_1m_1\tilde{m}_1}_{AB},X^{J_2m_2\tilde{m}_2}_{CD}][X^{AB}_{J_3m_3\tilde{m}_3},X^{CD}_{J_4m_4\tilde{m}_4}] \\
& -i\mathcal{D}_{Jm\tilde{m}\ J_1m_1\tilde{m}_1\rho_1\ J_2m_2\tilde{m}_2\rho_2}\partial_0 A_{J_1m_1\tilde{m}_1\rho_1}[B_{Jm\tilde{m}},A_{J_2m_2\tilde{m}_2\rho_2}] \\
& -\mu\sqrt{J_1(J_1+1)}\mathcal{D}_{J_2m_2\tilde{m}_2\ J_1m_1\tilde{m}_10\ Jm\tilde{m}\rho}B_{J_1m_1\tilde{m}_1}[A_{Jm\tilde{m}\rho},B_{J_2m_2\tilde{m}_2}] \\
& -\frac{1}{2}\mathcal{C}^{Jm\tilde{m}}_{J_1m_1\tilde{m}_1\ J_3m_3\tilde{m}_3}\mathcal{D}_{Jm\tilde{m}\ J_2m_2\tilde{m}_2\rho_2\ J_4m_4\tilde{m}_4\rho_4}[B_{J_1m_1\tilde{m}_1},A_{J_2m_2\tilde{m}_2\rho_2}] \\
& [B_{J_3m_3\tilde{m}_3},A_{J_4m_4\tilde{m}_4\rho_4}] \\
& -i\frac{\mu}{2}\rho_1(J_1+1)\mathcal{E}_{J_1m_1\tilde{m}_1\rho_1\ J_2m_2\tilde{m}_2\rho_2\ J_3m_3\tilde{m}_3\rho_3}A_{J_1m_1\tilde{m}_1\rho_1}[A_{J_2m_2\tilde{m}_2\rho_2},A_{J_3m_3\tilde{m}_3\rho_3}] \\
& +\frac{1}{4}\mathcal{D}^{Jm\tilde{m}}_{J_1m_1\tilde{m}_1\rho_1\ J_3m_3\tilde{m}_3\rho_3}\mathcal{D}_{Jm\tilde{m}\ J_2m_2\tilde{m}_2\rho_2\ J_4m_4\tilde{m}_4\rho_4}[A_{J_1m_1\tilde{m}_1\rho_1},A_{J_2m_2\tilde{m}_2\rho_2}] \\
& [A_{J_3m_3\tilde{m}_3\rho_3},A_{J_4m_4\tilde{m}_4\rho_4}] \\
& +\mathcal{F}^{J_1m_1\tilde{m}_1\kappa_1}_{J_2m_2\tilde{m}_2\kappa_2\ Jm\tilde{m}}\Psi^{\dagger}_{AJ_1m_1\tilde{m}_1\kappa_1}[B_{Jm\tilde{m}},\Psi^{A}_{J_2m_2\tilde{m}_2\kappa_2}] \\
& +\mathcal{G}^{J_1m_1\tilde{m}_1\kappa_1}_{J_2m_2\tilde{m}_2\kappa_2\ Jm\tilde{m}\rho}\Psi^{\dagger}_{AJ_1m_1\tilde{m}_1\kappa_1}[A_{Jm\tilde{m}\rho},\Psi^{A}_{J_2m_2\tilde{m}_2\kappa_2}] \\
& -i(-1)^{m_2-\tilde{m}_2+\frac{\kappa_2}{2}}\mathcal{F}^{J_1m_1\tilde{m}_1\kappa_1}_{J_2-m_2\tilde{m}_2\kappa_2\ Jm\tilde{m}}\Psi^{\dagger}_{AJ_1m_1\tilde{m}_1\kappa_1}[X^{AB}_{Jm\tilde{m}},\Psi^{\dagger}_{BJ_2m_2\tilde{m}_2\kappa_2}] \\
& +i(-1)^{-m_1+\tilde{m}_1+\frac{\kappa_1}{2}}\mathcal{F}^{J_1-m_1\tilde{m}_1\kappa_1}_{J_2m_2\tilde{m}_2\kappa_2\ Jm\tilde{m}}\Psi^{A}_{J_1m_1\tilde{m}_1\kappa_1}[X^{Jm\tilde{m}}_{AB},\Psi^{B}_{J_2m_2\tilde{m}_2\kappa_2}]\Big\},
\end{aligned} \tag{G.1}$$

where the summation over the indices that appear twice or more than twice in $S^{int}_{R\times S^3/Z_k}$ is assumed and $\tilde{m}$ only takes $\frac{k}{2}n$ $(n\in\boldsymbol{Z})$.

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